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From an adhesive to a brittle delamination model in thermo-visco-elasticity

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Abstract

We address the analysis of a model for *brittle delamination* of two visco-elastic bodies, bonded along a prescribed surface. The model also encompasses thermal effects in the bulk. The related PDE system for the displacements, the absolute temperature, and the delamination variable has a highly nonlinear character. On the contact surface, it features frictionless Signorini conditions and a nonconvex, brittle constraint acting as a transmission condition for the displacements.

We prove the existence of (weak/energetic) solutions to the associated Cauchy problem, by approximating it in two steps with suitably regularized problems. We perform the two consecutive passages to the limit via refined variational convergence techniques.

1 Introduction

This paper deals with the analysis of a model describing the evolution of brittle delamination between two visco-elastic bodies Ω_+ and Ω_-, bonded along a *prescribed* contact surface Γ_C, over a fixed time interval $(0,T)$. The modeling of delamination follows the approach by M. Frémond [Fré88, Fré02], which treats this phenomenon within the class of generalized standard materials [HN75]. More precisely, the adhesiveness of the bonding is modeled with the aid of an internal variable, the so-called delamination variable $z : (0,T)\times\Gamma_C \to [0,1]$, which describes the fraction of fully effective molecular links in the bonding. Hence, $z(t,x)=1$ means that the bonding at time $t\in(0,T)$ is fully intact in the material point $x\in\Gamma_C$, whereas for $z(t,x)=0$ the bonding is completely broken. The weakening of the bonding is a dissipative and unidirectional process, which is assumed to be rate-independent. These facts are modeled by the positively 1-homogeneous dissipation potential

$$\mathcal{R}_1(\dot z) := \int_{\Gamma_C} \mathrm{R}_1(\dot z)\,\mathrm{d}S \qquad \text{with} \quad \mathrm{R}_1(\dot z) := \begin{cases} a_1|\dot z| & \text{if } \dot z \le 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.1a}$$

where $\dot z$ is the partial time-derivative of z. A further dissipative process is due to viscosity in the bulk and the amount of dissipated energy is described by the positively 2-homogeneous dissipation potential

$$\mathcal{R}_2(\dot e) := \int_{\Omega\setminus\Gamma_C} \mathrm{R}_2(\dot e)\,\mathrm{d}x \qquad \text{with} \quad \mathrm{R}_2(\dot e) := \tfrac{1}{2}\dot e : \mathbb{D} : \dot e \tag{1.1b}$$

acting on the rate of the linearized strain tensor e. Here, $\Omega = \Omega_-\cup\Gamma_C\cup\Omega_+$ and $\mathbb{D}$ is a positively definite, symmetric fourth order tensor. In particular, the specific dissipation rate $\mathrm{R}(\dot e,\dot z) = \mathrm{R}_2(\dot e)\mathrm{d}x + \mathrm{R}_1(\dot z)\mathrm{d}S$ is in general a measure which reflects the mixed (i.e., rate-dependent and rate-independent) character of the model. Its absolutely continuous part is given by the (pseudo)potential of *viscous-type* dissipative forces in the bulk. The possibly concentrating part, supported on Γ_C, features the *rate-independent* dissipation metric R_1.

The visco-elastic response in the bulk material is modeled with the aid of Kelvin-Voigt rheology. This rheological model can be described by a parallel arrangement of a linear spring, which instantaneously produces a deformation in proportion to a load, and a dashpot, which instantaneously produces a velocity in proportion to a load. In other words, in a Kelvin-Voigt visco-elastic solid, a sudden application of a load will not cause an immediate deflection, since it is damped (cf. dashpot arranged in parallel with the spring). Instead, a deformation is built up rather gradually. Hence, the stress tensor of a Kelvin-Voigt visco-elastic solid is of the form $\sigma = \mathbb{C} : e + \mathrm{DR}_2(\dot e)$, where $\mathbb{C}$ is a symmetric, positive definite fourth order tensor and DR_2 is the derivative of the viscous dissipation density R_2; hereafter, with D we will denote the Gâteau derivative. For more details on the rheological modeling of visco-elastic solids the reader is referred to e.g. [Fun65].

As a further constitutive property of the bulk material it is assumed, that temperature changes cause additional stresses due to thermal expansion. Following [Rou10], for the stress tensor including visco-elastic response and thermal expansion stresses we use the ansatz

$$\sigma(e,\dot e,\theta) := \mathbb{C} : e + \mathrm{DR}_2(\dot e) - \theta\mathbb{C} : \mathbb{E} \tag{1.2}$$

with $\theta > 0$ the absolute temperature and $\mathbb{E}$ the symmetric matrix of thermal expansion coefficients.

The unknown states in our model are given by the displacement field $u : (0,T) \times \Omega_- \cup \Omega_+ \to \mathbb{R}^d$, the delamination variable $z : (0,T) \times \Gamma_\mathrm{C} \to [0,1]$ and the absolute temperature $\theta : (0,T) \times \Omega \to (0,\infty)$. The PDE system describing their evolution consists of the momentum balance for u, the heat equation for θ and a flow rule for z, which couple the three unknowns in a highly nonlinear manner, see Section 2.1. In the analysis, however, we will treat a weak formulation of this PDE system, the so-called *energetic formulation*. This notion stems from the fact that this formulation involves the energy and dissipation functionals related to the PDE system.

For the delamination system the overall Helmholtz *free energy* $\Psi = \Psi(u,z,\theta)$ consists of a bulk and of a surface contribution

$$\Psi(u,z,\theta) = \Psi^{\mathrm{bulk}}(u,z,\theta) + \Psi^{\mathrm{surf}}(u,z,\theta), \tag{1.3}$$

where $\Psi^{\mathrm{bulk}}(u,z,\theta) = \int_{\Omega\backslash\Gamma_\mathrm{C}} \mathrm{W}(e(u),\theta)\,\mathrm{d}x$ with $W(e,\theta) := \frac{1}{2}e : \sigma(e,\dot{e},\theta) - \psi_0(\theta)$. Here, σ is defined by (1.2) and $\psi_0 : (0,+\infty) \to \mathbb{R}$ is a strictly convex function. While $-\psi_0(\theta)$ is the (purely) *thermal* part of the free energy, the (purely) *mechanical* part is given by the functional

$$\Phi(u,z) := \Phi^{\mathrm{bulk}}(u) + \Phi^{\mathrm{surf}}(\llbracket u \rrbracket, z)\,, \tag{1.4}$$

where

$$\Phi^{\mathrm{bulk}}(u) = \int_{\Omega\backslash\Gamma_\mathrm{C}} \tfrac{1}{2}\, e : \mathbb{C} : e\,\mathrm{d}x \quad \text{and} \quad \Phi^{\mathrm{surf}}(\llbracket u \rrbracket, z) = \int_{\Gamma_\mathrm{C}} \phi^{\mathrm{surf}}(\llbracket u \rrbracket, z)\,\mathrm{d}S. \tag{1.5}$$

At fixed temperature, for fully rate-independent systems the energetic formulation was developed in [Mie05, MT04, FM06], and in this setting it is solely defined by the global *stability condition* and the global *energy balance*, i.e. $(u,z) : (0,T) \to \mathcal{Q}$ is an energetic solution of the rate-independent system $(\mathcal{Q}, \Phi, \mathcal{R}_1)$, given by a state space $\mathcal{Q}$, the energy functional Φ and the dissipation potential $\mathcal{R}_1$, if for all $t \in (0,T)$:

$$\forall\,(\tilde{u},\tilde{z}) \in \mathcal{Q} : \quad \Phi(t,u(t),z(t)) \le \Phi(t,\tilde{u},\tilde{z}) + \mathcal{R}_1(\tilde{z} - z(t)) \qquad \text{(stability)}\,, \tag{1.6a}$$

$$\Phi(t,u(t),z(t)) + \mathcal{R}_1(z(t) - z(0)) = \Phi(0,u(0),z(0)) + \int_0^t \partial_t\Phi(s,u(s),z(s))\,\mathrm{d}s \quad \text{(energy balance)}\,. \tag{1.6b}$$

However, conditions (1.6) do not supply a suitable energetic formulation in the temperature-dependent, viscous setting. For this context, a suitable notion was introduced in [Rou10], see Definition 3.3 here. Instead of the two conditions (1.6), the energetic formulation in the temperature-dependent, viscous setting consists of four conditions: a weak formulation of the momentum balance for u, a weak formulation of the heat equation for θ, a so-called *semistability* condition for z and an energy (in-)equality. The latter two conditions correspond to those in (1.6). In particular, the notion of *semistability* highlights a significant difference, as, here, stability is only tested for z, while $\tilde{u}$ is kept fixed as a solution u, i.e.

$$\forall\, t \in (0,T)\ \forall\,\text{test functions } \tilde{z} : \quad \Phi(t,u(t),z(t)) \le \Phi(t,u(t),\tilde{z}) + \mathcal{R}_1(\tilde{z} - z(t)) \qquad \text{(semistability)}\,. \tag{1.7}$$

The analogous minimality test for u with fixed solution z corresponds to the weak formulation of the momentum balance. Hence, we want to stress that the energetic formulation in the sense of [Rou10] splits the rate-independent stability condition (1.6a) into two separate conditions: the weak momentum balance for u (with z fixed) and the semistability for z (with u fixed).

The adapted energetic formulation of Definition 3.3 will be analyzed for our delamination model in visco-elastic solids with thermal effects. In particular, we aim at a model for *brittle* delamination, i.e. it involves the

$$\text{brittle constraint:} \qquad z\llbracket u \rrbracket = 0 \quad \text{a.e. on } (0,T) \times \Gamma_\mathrm{C}\,, \tag{1.8}$$

where $\llbracket u \rrbracket$ is the jump of u across Γ_C. This condition allows for displacement jumps only in points $x \in \Gamma_\mathrm{C}$, where the bonding is completely broken, i.e. $z(t,x) = 0$; in points where $z(t,x) > 0$ it ensures $\llbracket u \rrbracket = 0$, i.e. the continuity of the displacements. In other words, the brittle constraint (1.8) distinguishes between the crack set where the displacements may jump and the complementary set with active bonding, where it imposes a transmission condition on the displacements. Moreover, our model contains a non-penetration constraint ensuring that the two parts of the body, Ω_- and Ω_+, cannot interpenetrate along Γ_C:

$$\text{non-penetration condition:} \qquad [\![u]\!] \cdot \mathrm{n} \geq 0 \quad \text{a.e. on } (0,T) \times \Gamma_C \,. \tag{1.9}$$

Here, n denotes the unit normal to Γ_C oriented from Ω_+ to Ω_-.

The extremely strict and nonconvex brittle constraint (1.8) causes severe difficulties in the existence analysis, even in the fully rate-independent setting (with fixed temperature and no viscosity), which was addressed in [RSZ09]. Therein, the existence of energetic solutions in the sense of (1.6) was not proved directly, but by passing to the limit in a suitable approximation procedure, where (1.8) was replaced by the so-called *adhesive contact* condition. The latter model involves an energy term which penalizes displacement jumps in points with positive z, but does not strictly exclude them, i.e. the

$$\text{adhesive contact term:} \qquad \frac{k}{2} \int_{\Gamma_C} z \left| [\![u]\!] \right|^2 \mathrm{d}S \,. \tag{1.10}$$

The existence of energetic solutions for the related rate-independent system was proved in [KMR06]. As $k \to \infty$ it was shown in [RSZ09] that the (fully) rate-independent systems of adhesive contact approximate the system for brittle delamination in the sense of Γ-convergence of rate-independent processes developed in [MRS08].

Our aim is to apply a similar strategy in the viscous, temperature-dependent setting. For this, we want to make use of the results in [RR11b] (see also [RR11a]), where the existence of energetic solutions in the sense of [Rou10] was proved for adhesive contact in visco-elastic materials with thermal effects. However, as this notion of solution splits the stability test into two separate conditions, weak momentum balance for u and semistability for z, we are not able to perform the limit passage $k \to \infty$ in the model from [RR11b] without adding suitable regularization terms. These will allow us to gain additional information which, in turn, enables us to construct test functions for the semistability condition and the momentum balance suitably fitted to the properties of the solutions.

We postpone a thorough discussion of these regularization terms to Section 2, where we gain further insight into the PDE system, reveal its analytical difficulties, and explain our results. At this point, let us just mention that our regularizations will consist of a gradient term for z and of a term of p-growth in the strain e, with p larger than space dimension, ensuring the continuity of the displacements in each of the subdomains Ω_- and Ω_+. It was proved in [MRT12] that the model for brittle delamination (without a gradient of z), also treated in [RSZ09], describes the evolution of a Griffith-crack along Γ_C. This means that $z \in \{0,1\}$, only, and hence z marks the crack set and the unbroken part of Γ_C. The fully rate-independent brittle delamination model treated in [MRT12, Tho08, RSZ09] therefore complies with crack models treated in e.g. [BFM08, DMFT05], but on a *prescribed interface*, see also [NO08, KMZ08]. In the visco-elastic, temperature-dependent setting we also want to ensure that $z \in \{0,1\}$ and therefore we choose the regularization such that z is the indicator function of a set of finite perimeter in Γ_C. As the perimeter is a nonconvex term, not well-suited for the tools used to prove existence in [RR11b], we first approximate it by a Modica-Mortola term (2.13). Thus, our approximation procedure is the following:

1. From the model for adhesive contact with Modica-Mortola regularization (called *Modica-Mortola adhesive contact model*) we will pass to the model for adhesive contact with perimeter regularization (called SBV-*adhesive contact model*) in Section 4;

2. from the SBV-adhesive contact model we will then pass in Sections 5 and 6 to the SBV-*brittle delamination model* (i.e. the model which incorporates the brittle constraint (1.8), but still contains the perimeter term for the delamination variable $z \in \{0,1\}$), thus proving the main result of this paper, Thm. 5.1.

Crucial for the passage from adhesive contact to brittle delamination in the visco-elastic, temperature-dependent setting is the construction of suitable test functions for the momentum balance. This requires additional information on the semistable delamination variables which solve the adhesive problems. In fact, it involves a fine analysis of their properties based on tools of geometric measure theory, to which we have devoted the whole Section 6. Therein, it will be proved that delamination variables which are semistable for the adhesive or the brittle problems have the so-called *support property*, which excludes that their support contains isolated subsets of arbitrarily small Lebesgue-measure or arbitrarily *thin cusps*, see Definition 6.4. This property will be used to verify *support convergence*, which means that the supports of the delamination variables solving the SBV-adhesive contact problems can be enclosed into balls around the support of the delamination variable for the SBV-brittle delamination model, and the radii of these balls tend to 0.

This support convergence will be the key property to construct test functions suited for the limit passage in the weak momentum balance from adhesive to brittle. In this connection, let us mention that, in contrast to the fully rate-independent case treated in [RSZ09], pure Γ-convergence of the systems in the sense of [MRS08] is no longer sufficient for the present visco-elastic, temperature-dependent systems. Here, Mosco-convergence will be needed.

Plan of the paper. After further discussing and motivating our approximation of the brittle delamination model via the SBV-adhesive and the Modica-Mortola adhesive systems in Section 2, in Section 3 we will first collect all the assumptions on the domain and the given data. Hence, we will introduce the energetic formulation of the visco-elastic, temperature-dependent systems for adhesive contact and brittle delamination and finally discuss the general strategy for proving the existence of energetic solutions. In Section 4 we will carry out the limit passage from Modica-Mortola to SBV-adhesive contact, see Thm. 4.3, in order to obtain an existence result for the SBV-adhesive contact systems (Thm. 4.1). This analysis relies on the existence of*energetic solutions* to the Modica-Mortola adhesive contact system, stated in Thm. 4.2, which shall be obtained by passing to the limit in a suitable time-discretization scheme in Appendix A.1. These results will be used in order to prove our main result, Thm. 5.1, on the existence of energetic solutions for the SBV-brittle delamination systems. Indeed, in Section 5 we will pass with SBV-adhesive contact to SBV-brittle delamination. As mentioned above, this limit passage bears difficulties in the momentum balance, which can be solved by exploiting additional information on semistable delamination variables, viz. the *support property* and the *support convergence*. They will be proved in Section 6, by means of tools from geometric measure theory collected for the reader's convenience in Appendix A.2.

2 Analytical difficulties and our results

In this section, we first detail the *classical* formulation of the PDE system describing the brittle delamination model for visco-elastic materials with thermal effects. We then highlight the main difficulties related to its analysis and motivate its approximation by the SBV- and Modica-Mortola adhesive systems.

2.1 The classical formulation of the problem

Throughout the paper we assume that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain, with $\Omega = \Omega_+ \cup \Gamma_{\mathrm{C}} \cup \Omega_-$ and Γ_{C} representing the *prescribed* (flat) interface with possible delamination. We denote by n both the outward unit normal to $\partial\Omega$, and the unit normal to Γ_{C} oriented from Ω_+ to Ω_-. Given $v \in W^{1,2}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$, with v^+ (v^-) we signify the restriction of v to Ω_+ (Ω_-). We denote by

$$[\![v]\!] = v^+|_{\Gamma_{\mathrm{C}}} - v^-|_{\Gamma_{\mathrm{C}}} \ = \text{the jump of } v \text{ across } \Gamma_{\mathrm{C}}\,. \tag{2.1}$$

The PDE system, coupling the momentum equation in the bulk (2.2a) for the displacement u, the heat equation (2.2b) for the absolute temperature θ, and the evolution (2.2k)–(2.2n) for the delamination parameter z, formally reads:

$$
\begin{aligned}
& -\operatorname{div}\sigma(u,\dot u,\theta)=F && \text{in } (0,T)\times(\Omega\backslash\Gamma_{\rm C}), && (2.2a)\\
& c_{\rm v}(\theta)\dot\theta-\operatorname{div}\big(\mathbb{K}(e(u),\theta)\nabla\theta\big)=e(\dot u){:}\mathbb{D}{:}e(\dot u)-\theta\mathbb{E}{:}\mathbb{C}{:}e(\dot u)+H && \text{in } (0,T)\times(\Omega\backslash\Gamma_{\rm C}), && (2.2b)\\
& u=0 && \text{on } (0,T)\times\Gamma_{\rm D}, && (2.2c)\\
& \sigma(u,\dot u,\theta)\big|_{\Gamma_{\rm N}}\mathrm{n}=f && \text{on } (0,T)\times\Gamma_{\rm N}, && (2.2d)\\
& (\mathbb{K}(e(u),\theta)\nabla\theta)\mathrm{n}=h && \text{on } (0,T)\times\partial\Omega, && (2.2e)\\
& [\![\sigma]\!]\mathrm{n}=0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2f)\\
& [\![u]\!]\cdot\mathrm{n}\ge 0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2g)\\
& \sigma(u,\dot u,\theta)\big|_{\Gamma_{\rm C}}\mathrm{n}\cdot\mathrm{n}\ge 0 \quad \text{wherever } z(\cdot)=0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2h)\\
& \sigma(u,\dot u,\theta)\big|_{\Gamma_{\rm C}}\mathrm{n}\cdot[\![u]\!]=0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2i)\\
& z[\![u]\!]=0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2j)\\
& \dot z\le 0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2k)\\
& d\le a_1+a_0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2l)\\
& \dot z\,(d-a_0-a_1)=0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2m)\\
& d\in\partial I_{[0,1]}(z)+\partial_z J_\infty\big([\![u]\!],z\big) && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2n)\\
& \frac12\big(\mathbb{K}(e(u),\theta)\nabla\theta|_{\Gamma_{\rm C}}^{+}+\mathbb{K}(e(u),\theta)\nabla\theta|_{\Gamma_{\rm C}}^{-}\big)\cdot\mathrm{n}+\eta([\![u]\!],z)[\![\theta]\!]=0 && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2o)\\
& [\![\mathbb{K}(e(u),\theta)\nabla\theta]\!]\cdot\mathrm{n}=-a_1\dot z && \text{on } (0,T)\times\Gamma_{\rm C}, && (2.2p)
\end{aligned}
$$

where $\partial\Omega=\Gamma_{\rm D}\cup\Gamma_{\rm N}$ with $\Gamma_{\rm D}$ the *Dirichlet* and $\Gamma_{\rm N}$ the *Neumann* parts of the boundary $\partial\Omega$.

System (2.2) was derived in [RR11b, Sec. 2] starting from the Helmholtz free energy (1.3) and the dissipation potentials (1.1); its thermodynamical consistency was shown, in the sense that the Clausius-Duhem inequality and the positivity of temperature are satisfied. In the following lines, we will confine ourselves to just explaining the meaning of the equations; for more details we refer to [RR11b].

In (2.2a), (2.2d), (2.2f), (2.2h), and (2.2i), the term $\sigma:=\mathbb{D}{:}e(v)+\mathbb{C}{:}\big(e(u)-\mathbb{E}\theta\big)$ is the stress tensor, which encompasses Kelvin-Voigt rheology and thermal expansion, as explained along with (1.2). Here,

$$\mathbb{C},\,\mathbb{D}:\mathbb{R}^{d\times d}_{\rm sym}\to\mathbb{R}^{d\times d}_{\rm sym}\quad \text{are 4th-order positive definite and symmetric tensors, } \mathbb{C} \text{ potential,} \qquad (2.3)$$

viz. $\mathbb{C}_{ijkl}=\mathbb{C}_{jikl}=\mathbb{C}_{klij}$, and the same for $\mathbb{D}$, and the operator $\operatorname{div}\mathbb{C}{:}e(u)$ has a potential; $\mathbb{E}\in\mathbb{R}^{d\times d}$ is a matrix of thermal-expansion coefficients. Moreover, $F:(0,T)\times\Omega\to\mathbb{R}^d$ in (2.2a) is the applied bulk force, $f:(0,T)\times\Gamma_{\rm N}\to\mathbb{R}^d$ in (2.2d) is the applied traction, while $H:(0,T)\times\Omega\to\mathbb{R}$ in (2.2b) and $h:(0,T)\times\partial\Omega\to\mathbb{R}$ in (2.2e) are external heat sources.

In the heat equation (2.2b), the function $c_{\rm v}:(0,+\infty)\to(0,+\infty)$ is the *heat capacity* of the system. Moreover, $-\mathbb{K}(e,\theta)\nabla\theta$ determines the heat flux according to Fourier's law, with $\mathbb{K}=\mathbb{K}(e,\theta)$ as the positive definite matrix of heat conduction coefficients. The terms $e(\dot u){:}\mathrm{D}R_2(e(\dot u))=e(\dot u){:}\mathbb{D}{:}e(\dot u)$ and $-\theta\mathbb{E}{:}\mathbb{C}{:}e(\dot u)$ on the right-hand side of (2.2b) are heat sources due to viscous and thermal expansion stresses, and they generate a coupling between the heat and the momentum equation.

Further, (2.2c) and (2.2d) are the Dirichlet and Neumann conditions for u and (2.2e) is the Neumann condition for the heat flux across the interface; conditions (2.2g)–(2.2i) yield the complementarity form of the *Signorini contact conditions*, preventing penetration of either of the bodies Ω_+ and Ω_- along the interface. Furthermore, (2.2j) is the brittle constraint, which can be interpreted as a *transmission condition* on the contact surface $\Gamma_{\rm C}$, as explained along with (1.8).

The complementarity conditions (2.2k)–(2.2n) determine the evolution of the delamination variable. In particular, (2.2k) ensures the unidirectionality of the delamination process as crack healing is prevented. In (2.2l), (2.2m), the coefficient a_0 (resp. a_1) is the phenomenological specific energy per area which is stored (resp. dissipated) by disintegrating the adhesive. The overall activation energy to trigger the debonding process in the adhesive is then a_0+a_1. Moreover, in (2.2n), $\partial I_{[0,1]}$ denotes the subdifferential in the sense of convex analysis of the indicator function $I_{[0,1]}$ of the interval $[0,1]$. Here, $I_{[0,1]}$ is defined by $I_{[0,1]}(r) = 0$ if $r \in [0,1]$ and $I_{[0,1]}(r) = +\infty$ otherwise. The second operator in (2.2n) refers to the indicator function featuring the brittle constraint

$$J_\infty(v,z) = I_{\{vz=0\}}(v,z), \quad \text{i.e.} \quad J_\infty(v,z) = \begin{cases} 0 & \text{if } vz = 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Finally, conditions (2.2o) and (2.2p) balance the heat transfer across $\Gamma_{\rm C}$ with the ongoing crack growth. In particular, the function η in the boundary condition (2.2o) on $\Gamma_{\rm C}$ for θ is a heat-transfer coefficient, determining the heat convection through $\Gamma_{\rm C}$, which depends on the state of the bonding and on the distance between the crack tips. We refer to [RR11b, Rem. 3.3] for further details.

2.2 Regularization and approximation via adhesive contact models

The analysis of system (2.2) encounters several difficulties: first of all, the *mixed* character of the problem, coupling *rate-independent* evolution for z, with *rate-dependent* equations for u and θ. Let us also mention the highly nonlinear character of the heat equation, with a quadratic term on the right-hand side. The evolution of z is ruled by the complementarity conditions (2.2k)–(2.2n), which can be reformulated as the subdifferential inclusion

$$\partial I_{(-\infty,0]}(\dot z(t,x)) + \partial I_{[0,1]}(z(t,x)) + \partial_z J_\infty\big([\![u(t,x)]\!], z(t,x)\big) - a_0 - a_1 \ni 0, \quad (t,x) \in (0,T) \times \Gamma_{\rm C}, \tag{2.5}$$

where $\partial_z J_\infty$ denotes the (convex analysis) subdifferential of the brittle constraint J_∞ from (2.4) w.r.t. the variable z, viz.

$$\partial_z J_\infty\big([\![u]\!], z\big) = \begin{cases} \emptyset & \text{if } z \neq 0 \text{ and } [\![u]\!] \neq 0, \\ 0 & \text{if } [\![u]\!] = 0, \\ \mathbb{R} & \text{if } [\![u]\!] \neq 0 \text{ and } z = 0. \end{cases} \tag{2.6}$$

Let us observe that the subdifferential inclusion (2.5) for z is *triply nonlinear*, featuring three multivalued operators. An additional difficulty stems from the fact that the subdifferential of the brittle constraint J_∞ depends on $[\![u]\!]$, cf. (2.6).

Nonetheless, it is the analysis of the boundary value problem for the momentum equation which brings along the most challenging problems. Indeed, in view of (2.2g)–(2.2j), on the contact surface $\Gamma_{\rm C}$ we have for the displacement u a *double* constraint, namely the non-penetration $[\![u]\!] \cdot {\rm n} \geq 0$, and the *nonconvex* brittle constraint $z[\![u]\!] = 0$. Such constraints are reflected in the variational formulation of the boundary value problem for (2.2a) as a variational inequality, viz.

$$\begin{aligned} & [\![u]\!] \cdot {\rm n} \geq 0, \quad z[\![u]\!] = 0 \quad \text{on } (0,T) \times \Gamma_{\rm C}, \quad \text{and} \\ & \int_{\Omega\backslash\Gamma_{\rm C}} \big(\mathbb{D}{:}e(\dot u) + \mathbb{C}{:}(e(u) - \mathbb{E}\theta)\big) : e(v-u) \,{\rm d}x \geq \int_\Omega F \cdot (v{-}u) \,{\rm d}x + \int_{\Gamma_{\rm N}} f \cdot (v{-}u) \,{\rm d}x \end{aligned} \tag{2.7}$$

for all test functions v with suitable regularity and such that $[\![v]\!] \cdot {\rm n} \geq 0$ and $z[\![v]\!] = 0$ a.e. on $(0,T) \times \Gamma_{\rm C}$. A major difficulty is that the brittle constraint involves z, and accordingly the set of test functions in (2.7) depends on z.

The SBV-brittle delamination system. To handle the coupling of the brittle and of the non-penetration constraints, we will approximate system (2.2) by penalizing the condition $z[\![u]\!] = 0$ on $(0,T) \times \Gamma_{\rm C}$. For the passage to the limit in the weak formulation of the momentum equation, a suitable construction of approximate test functions will be needed. This construction relies on a higher spatial regularity for the displacement variable u. Therefore, we have to regularize the momentum equation (2.2a) by means of a *tensorial* p-Laplacian term, with $p > d$. More precisely, in the momentum balance (2.2a) and in the boundary conditions (2.2d), (2.2f), (2.2h), and (2.2i), from now on the stress tensor σ will be given by

$$\sigma = \sigma(u,v,\theta) := \mathbb{D}{:}e(v) + \mathbb{C}{:}\big(e(u){-}\mathbb{E}\theta\big) + \mathbb{H}{:}|e(u)|^{p-2}e(u) \qquad \text{with } p > d \tag{2.8}$$

and $\mathbb{H}$ a fourth-order symmetric positive-definite tensor. Note that the term $\mathbb{H}{:}|e(u)|^{p-2}e(u)$ ensures that $u \in W^{1,p}(\Omega_\pm) \subset \mathrm{C}^0(\overline{\Omega_\pm})$ (since $p > d$), which is crucial for tackling the brittle constraint $z[\![u]\!] = 0$.

Furthermore, we shall also regularize the delamination variable z through an additional gradient term $\mathcal{G}(z)$. Gradient regularizations of the type $\mathcal{G}(z) = \int_\Omega \frac{1}{r}|\nabla z|^r \,\mathrm{d}x$ are often used in models for damage (see e.g. [FN96, BSS05, MR06, TM10, Tho10, MRT12]), and adhesive contact [BBR08, BBR09]. Here, for technical reasons which will become apparent later on, we resort to a BV-type regularization, and hence take the state space $\mathcal{Z}$ for z as a subset of the space $\mathrm{BV}(\Gamma_\mathrm{C})$ of functions on bounded variation on Γ_C, whose distributional gradient is a finite Radon measure on Γ_C. Hence, we consider $\mathcal{G}_\mathrm{b}(z) = \mathrm{b}|\mathrm{D}z|(\Gamma_\mathrm{C})$ for some $\mathrm{b} > 0$, where $|\mathrm{D}z|(\Gamma_\mathrm{C})$ denotes the variation of the measure $\mathrm{D}z$ in Γ_C. Moreover, we add a further constraint in our delamination system, namely that the variable z only takes the values $\{0,1\}$. Therefore, our model accounts for just two states of the bonding between Ω_+ and Ω_-, viz. fully effective and completely ineffective. On the one hand, the feature that $z \in \{0,1\}$ makes our model closer to a Griffith-type model for crack evolution (along a *prescribed* interface). Therein, the delamination variable z individuates the crack set, and thus only takes either the value 0, or 1, see [Tho10, MRT12]. On the other hand, such a restriction brings along some analytical advantages, as the considerations in Sec. 6 will show later on. Since $z \in \{0,1\}$, it can be viewed as the characteristic function of a set Z with finite perimeter. Therefore, the gradient term $\mathcal{G}_\mathrm{b}$ reduces to

$$\mathcal{G}_\mathrm{b}(z) = \mathrm{b}|\mathrm{D}z|(\Gamma_\mathrm{C}) = \mathrm{b}P(Z,\Gamma_\mathrm{C}), \tag{2.9}$$

$P(Z,\Gamma_\mathrm{C})$ as the perimeter of the set Z in Γ_C, cf. Def. A.7. We will also use that $\mathcal{G}_\mathrm{b}(z) = \mathcal{H}^{d-2}(\mathrm{J}_z)$, where $\mathcal{H}^{d-2}$ denotes the $(d-2)$-dimensional Hausdorff measure and J_z is the jump set of $z \in \mathrm{SBV}(\Gamma_\mathrm{C};\{0,1\})$, see Def. A.13. Here, $\mathrm{SBV}(\Gamma_\mathrm{C};\{0,1\})$ is the set of characteristic functions of subsets of Γ_C with finite perimeter. In particular, the acronym SBV stands for *special functions of bounded variation*, which is the subspace of BV of functions whose total variation has no Cantor part, see [AFP05] for more details. With the regularization $\mathcal{G}_\mathrm{b}$ given by (2.9), the subdifferential inclusion (2.5) is *formally* replaced by

$$\partial I_{(-\infty,0]}(\dot z(t,x)) + \partial I_{[0,1]}(z(t,x)) + \partial_z J_\infty\big([\![u(t,x)]\!], z(t,x)\big) + \partial\mathcal{G}_\mathrm{b}(z(t,x)) - a_0 - a_1 \ni 0, \tag{2.10}$$

for a.a. $(t,x) \in (0,T) \times \Gamma_\mathrm{C}$. In fact, we will analyze a *weak* formulation of (2.10).

Throughout the paper, we shall refer to the PDE system (2.2), with (2.5) replaced by (2.10), and the stress σ given by (2.8), as the SBV-*brittle delamination system*. In fact, we shall propose a suitable notion of weak solution for this system, cf. Def. 3.9 of *energetic solution*. This solution concept consists of the weak formulations of the boundary-value problems for the momentum equation (2.2a) with σ from (2.8), for the enthalpy equation (2.2b), of a *semistability* condition in place of (2.10), and of an *energy (in-)equality*. Our main Theorem 5.1 states the existence of *energetic solutions* to the SBV-*brittle delamination system*. In what follows, we hint at the strategy for the proof of this existence result, and in doing so we motivate the two aforementioned gradient regularizations.

The SBV-adhesive contact system. In order to deal with the brittle constraint $z[\![u]\!]=0$ on $(0,T)\times\Gamma_\mathrm{C}$, we approximate problem (2.2), with an *adhesive contact* problem, where (2.2g)–(2.2i) are replaced by

$$\left.\begin{aligned} &[\![u(t,x)]\!]\cdot \mathrm{n} \ge 0\\ &\Big(\sigma(u(t,x),\dot u(t,x),\theta(t,x))\big|_{\Gamma_\mathrm{C}}\mathrm{n}+kz(t,x)[\![u(t,x)]\!]\Big)\cdot\mathrm{n} \ge 0\\ &\Big(\sigma(u(t,x),\dot u(t,x),\theta(t,x))\big|_{\Gamma_\mathrm{C}}\mathrm{n}+kz(t,x)[\![u(t,x)]\!]\Big)\cdot[\![u(t,x)]\!] = 0 \end{aligned}\right\} \tag{2.11}$$

for a.a. $(t,x) \in (0,T) \times \Gamma_\mathrm{C}$, whereas instead of (2.10) we have

$$\partial I_{(-\infty,0]}(\dot z(t,x)) + \partial I_{[0,1]}(z(t,x)) + \frac{1}{2}k\big|[\![u(t,x)]\!]\big|^2 + \partial\mathcal{G}_\mathrm{b}(z(t,x)) - a_0 - a_1 \ni 0, \quad (t,x) \in (0,T)\times\Gamma_\mathrm{C}, \tag{2.12}$$

with $k > 0$ a fixed constant. Formally, (2.5), along with the brittle constraint $z[\![u]\!] = 0$ on $(0,T)\times\Gamma_\mathrm{C}$, arises in the limit as $k \to \infty$ of (2.11) and (2.12). In [RSZ09], it was proved for the *isothermal*, purely *rate-independent* delamination model, that the *brittle delamination* system can be obtained by passing to the limit as $k \to \infty$ in its *adhesive contact* approximation. We shall perform the limit $k \to \infty$ for our own temperature-dependent model as well. In doing so, the most delicate step is the passage to the limit in the momentum equation (2.2a). For this, we need to construct special test functions for its weak formulation (cf. (2.7)). Such a construction turns out to be a highly nontrivial technical point. To carry it out, we need the mentioned p-Laplacian regularization of the displacement variable u in (2.2a). We also rely on the gradient regularization $\mathcal{G}_\mathrm{b}$ (2.9) for z. Indeed, such a term guarantees the strong convergence in $L^q((0,T)\times\Gamma_\mathrm{C})$ for every $1 \le q < \infty$ for the approximate sequence $(z_k)_k$, in addition to the weak* convergence in $L^\infty((0,T)\times\Gamma_\mathrm{C})$. We shall refer to the approximate problem obtained replacing (2.2g)–(2.2j) and (2.10), with (2.11) and (2.12), respectively (combined with the quasi-static momentum equation (2.2a) with σ from (2.8)), as the SBV-*adhesive contact system.* First, we shall prove existence of energetic solutions for the related Cauchy problem in Theorem 4.1. Hence we shall take the limit as $k \to \infty$: Thm. 5.1 states that, up to a subsequence, solutions to the SBV-adhesive contact systems converge to solutions of the SBV-*brittle* delamination system.

The Modica-Mortola adhesive contact system. Since the SBV-gradient term in (2.12) is *nonconvex*, to prove existence for the (weak formulation of the) SBV-adhesive system we use a Modica-Mortola type approximation. This kind of regularization has been well known in the mathematical literature for more than thirty years. Indeed, in the papers [MM77, Mod87] (see also [Alb00]), within phase transition modeling it was proved that the so-called static Modica-Mortola functional Γ-converges to the static perimeter functional. Modica-Mortola approximations in the context of models for volume damage have also been exploited in [Gia05, Tho11]. The Modica-Mortola functional is

$$\mathcal{G}_m(z) := \int_{\Gamma_\mathrm{C}} \left(m g(z) + \frac{1}{2m}|\nabla z|^2 + I_{[0,1]}(z) \right) \mathrm{d}S \qquad \text{with } g(z) = z^2(1-z)^2 \text{ and } m > 0. \tag{2.13}$$

Accordingly, we will approximate the SBV-adhesive system by replacing the subdifferential inclusion (2.12) for z, with

$$\partial I_{(-\infty,0]}(\dot{z}(t,x)) + \partial I_{[0,1]}(z(t,x)) + \frac{1}{2}k\big|[\![u(t,x)]\!]\big|^2 + m g'(z(t,x)) - \frac{1}{m}\Delta z(t,x) - a_0 - a_1 \ni 0, \tag{2.14}$$

for a.a. $(t,x) \in (0,T)\times\Gamma_\mathrm{C}$. The resulting approximate problem will be called *Modica-Mortola adhesive contact system.*

3 General setup and weak formulation

In this section we present a suitable notion of weak formulation for the visco-elastic, temperature-dependent systems of adhesive contact and brittle delamination, i.e. the *energetic formulation* developed in [Rou10]. Prior to establishing this formulation in Section 3.3, in Sec. 3.1 we perform the so-called *enthalpy reformulation* of system (2.2) (and its regularizations), first introduced in [Rou10]. Then, in Section 3.4 the general strategy of the existence proof will be outlined. Although Definition 3.3 of energetic solution does not rely on a specific set of assumptions on the geometrical setting and the problem data, subsequent results such as Thm. 3.1 do. That is why, we have chosen to preliminarily collect all of the assumptions on the given data in Section 3.2, appropriate for all the systems and all the limit passages. Let us now fix some general notation.

Notation 3.1 (Function spaces) Throughout the paper, for $p \in (1,\infty)$ we shall adopt the notation

$$W^{1,p}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d) = \left\{ v \in W^{1,p}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d) \ : \ \ v = 0 \ \text{ on } \Gamma_\mathrm{D} \right\}. \tag{3.1}$$

We recall that

$$u \mapsto u|_\Gamma : W^{1,p}(\Omega\backslash\Gamma_\mathrm{C}) \to W^{1,1-\frac{1}{p}}(\Gamma) \text{ continuously} \tag{3.2}$$

with $\Gamma = \partial\Omega$, or $\Gamma = \Gamma_\mathrm{C}$, or $\Gamma = \Gamma_\mathrm{N}$. Furthermore, we shall exploit that, for $p > d$, the following embedding holds for $W^{1,p}(\Omega_\pm)$ (and obviously for the Sobolev space $W^{1,p}(\Omega_\pm;\mathbb{R}^d)$ of vector-valued functions)

$$W^{1,p}(\Omega_\pm) \subset \mathrm{C}^0(\overline{\Omega_\pm}) \text{ compactly.} \tag{3.3}$$

We shall denote by $\langle \cdot, \cdot \rangle$ the duality pairing between the spaces $W^{1,q}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*$ and $W^{1,q}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ and $W^{1,q}(\Omega\backslash\Gamma_{\rm C})^*$ and $W^{1,q}(\Omega\backslash\Gamma_{\rm C})$ for any $1 \leq q < \infty$.

For a (separable) Banach space X, we shall use the notation $\mathrm{BV}([0,T];X)$ for the space of functions from $[0,T]$ with values in X that have bounded variation on $[0,T]$. Notice that all these functions are defined everywhere on $[0,T]$.

Finally, throughout the paper we will use the symbols c, c', C, and C', for various positive constants depending only on known quantities.

3.1 Enthalpy reformulation

Following [Rou10, RR11b], we shall in fact analyze a reformulation of the PDE system (2.2), in which we replace the heat equation (2.2b) with an *enthalpy* equation, cf. system (3.6) below. This is motivated by the fact that the nonlinear term $c_{\rm V}(\theta)\dot\theta$ makes it difficult to implement a time-discretization scheme for (2.2b). In turn, time-discretization will provide the basic existence result for the Modica-Mortola adhesive contact system. Therefore, following [Rou10, RR11b] we are going to resort to a change of variables for θ, by means of which $c_{\rm V}(\theta)\dot\theta$ is replaced by the *linear* contribution $\dot w$.

Hereafter, we switch from the absolute temperature θ, to the enthalpy w, defined via the so-called *enthalpy transformation*, viz.

$$w = h(\theta) := \int_0^\theta c_{\rm V}(r)\,\mathrm{d}r. \tag{3.4}$$

Thus, h is a primitive function of $c_{\rm V}$, normalized in such a way that $h(0) = 0$. Since $c_{\rm V}$ is strictly positive (cf. assumption (3.8a) later on), h is strictly increasing. Thus, we are entitled to define

$$\Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \qquad \mathcal{K}(e,w) := \frac{\mathbb{K}(e,\Theta(w))}{c_{\rm V}(\Theta(w))}, \tag{3.5}$$

where h^{-1} here denotes the inverse function to h. With transformation (3.4) and (3.5), the classical formulation (2.2) of the SBV-*brittle* delamination system (with σ from (2.8) and the additional SBV-gradient regularization in (2.10)), turns into

$$-\operatorname{div}\big(\mathrm{DR}_2(e(\dot u)) + \mathrm{DW}_2(e(u)) - \mathbb{B}\Theta(w)) + \mathrm{DW}_p(e(u))\big) = F \qquad \text{in } (0,T)\times(\Omega\backslash\Gamma_{\rm C}), \tag{3.6a}$$

$$\dot w - \operatorname{div}\big(\mathcal{K}(e(u),w)\nabla w\big) = -\Theta(w)\mathbb{B}{:}e(\dot u) + H \qquad \text{in } (0,T)\times(\Omega\backslash\Gamma_{\rm C}), \tag{3.6b}$$

$$u = 0 \qquad \text{on } (0,T)\times\Gamma_{\rm D}, \tag{3.6c}$$

$$\sigma(u,\dot u,w)\big|_{\Gamma_{\rm N}}\mathrm{n} = f \qquad \text{on } (0,T)\times\Gamma_{\rm N}, \tag{3.6d}$$

$$(\mathcal{K}(e(u),w)\nabla w)\mathrm{n} = h \qquad \text{on } (0,T)\times\partial\Omega, \tag{3.6e}$$

$$[\![\sigma(u,\dot u,w)]\!]\mathrm{n} = 0 \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6f}$$

$$[\![u]\!]\cdot\mathrm{n} \geq 0 \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6g}$$

$$\sigma(u,\dot u,w)\big|_{\Gamma_{\rm C}}\mathrm{n}\cdot\mathrm{n} \geq 0 \quad \text{wherever } z(\cdot) = 0 \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6h}$$

$$\sigma(u,\dot u,w)\big|_{\Gamma_{\rm C}}\mathrm{n}\cdot[\![u]\!] = 0 \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6i}$$

$$\partial I_{(-\infty,0]}(\dot z) + \partial I_{[0,1]}(z) + {+}\partial_z J_\infty\big([\![u]\!],z\big) + \partial\mathcal{G}_{\rm b}(z) - a_0 - a_1 \ni 0 \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6j}$$

$$\frac{1}{2}\big(\mathcal{K}(e(u),w)\nabla w|^+_{\Gamma_{\rm C}} + \mathcal{K}(e(u),w)\nabla w|^-_{\Gamma_{\rm C}}\big)\cdot\mathrm{n} + \eta([\![u]\!],z)[\![\Theta(w)]\!] = 0 \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6k}$$

$$[\![\mathcal{K}(e(u),w)\nabla w]\!]\cdot\mathrm{n} = -a_1\dot z \qquad \text{on } (0,T)\times\Gamma_{\rm C}, \tag{3.6l}$$

where $\mathrm{W}_n(e) := \frac{1}{n}|e|^n$ with $n \in \{2,p\}$ in (3.6a) and, where we have introduced the placeholder

$$\mathbb{B} := \mathbb{C}{:}\mathbb{E}\,.$$

Furthermore, in the momentum equation and in the enthalpy equation, we have incorporated the notation from (1.1) for the dissipation potentials. With slight abuse, we also write

$$\sigma(u,v,w) := \sigma(u,v,\Theta(w)) = \big[\mathrm{DR}_2(e(v)) + \mathrm{DW}_2(e(u)) - \mathbb{B}\Theta(w)) + \mathrm{DW}_p(e(u))\big].$$

With obvious changes, one also obtains the classical *enthalpy* reformulation of the SBV-*adhesive* (cf. (2.11) and (2.12)) and of the *Modica-Mortola* adhesive (cf. (2.14)) contact systems.

3.2 Assumptions on the domain and the given data

Assumptions on the reference domain Ω. We suppose that

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is bounded, Ω_-, Ω_+, Ω are Lipschitz, $\Omega_+ \cap \Omega_- = \emptyset$, (3.7a)
- $\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N}$, $\Gamma_{\rm D}$, $\Gamma_{\rm N}$ open subsets in $\partial\Omega$, $\Gamma_{\rm D} \cap \Gamma_{\rm N} = \emptyset$, $\mathcal{H}^{d-1}(\Gamma_{\rm D}) > 0$, (3.7b)
- $\Gamma_{\rm C} \subset \mathbb{R}^{d-1}$ is *flat*, i.e. it is a subset of a hyperplane of $\mathbb{R}^d$ and $\mathcal{H}^{d-1}(\Gamma_{\rm C}) = \mathcal{L}^{d-1}(\Gamma_{\rm C}) > 0$, (3.7c)

where $\mathcal{H}^{d-1}$ and $\mathcal{L}^{d-1}$ respectively denote the $(d-1)$-dimensional Hausdorff and Lebesgue measures.
Assumptions on the given data. We impose the following conditions on $c_{\rm V}$, $\mathbb{K}$, and η:

$$c_{\rm V} : [0, +\infty) \to \mathbb{R}^+ \quad \text{continuous,} \tag{3.8a}$$

$$\exists\, \omega_1 \geq \omega > \frac{2d}{d{+}2},\ c_1 \geq c_0 > 0\ \forall \theta \in \mathbb{R}^+ : \quad c_0(1{+}\theta)^{\omega-1} \leq c_{\rm V}(\theta) \leq c_1(1{+}\theta)^{\omega_1-1}, \tag{3.8b}$$

$$\mathbb{K} : \mathbb{R}^{d\times d} \times \mathbb{R} \to \mathbb{R}^{d\times d} \text{ is bounded, continuous, and} \tag{3.8c}$$

$$\inf_{(e,w,\xi)\in\mathbb{R}^{d\times d}_{\rm sym}\times\mathbb{R}\times\mathbb{R}^d,\ |\xi|=1} \mathcal{K}(e, w)\xi{:}\xi = \mathsf{k} > 0, \tag{3.8d}$$

and that

$$\begin{array}{l} \eta : \Gamma_{\rm C}\times \big(\mathbb{R}^d \times \mathbb{R}\big) \to \mathbb{R}^+ \quad \text{is a Carathéodory function such that} \\ \exists\, C_\eta > 0\ \exists\, \sigma_1,\, \sigma_2 > 0 \text{ such that } \forall\, (x, v, z) \in \Gamma_{\rm C}{\times}\mathbb{R}^d \times \mathbb{R} : \quad |\eta(x, v, z)| \leq C_\eta(1{+}|v|^{\sigma_1}{+}|z|^{\sigma_2}). \end{array} \tag{3.8e}$$

In particular, notice that any *polynomial growth* of η w.r.t. the variables (v, z) is admissible.

Remark 3.2 It is immediate to deduce from (3.8b) that

$$\exists\, C_\theta^1,\, C_\theta^2 > 0 \quad \forall\, w \in \mathbb{R}^+ : \quad C_\theta^1(w^{1/\omega_1} - 1) \leq \Theta(w) \leq C_\theta^2(w^{1/\omega} + 1)\,. \tag{3.9}$$

Moreover, it follows from (3.8b)–(3.8c) and the definition (3.5) of $\mathcal{K}$ that

$$\exists\, C_{\mathcal{K}} > 0 \quad \forall \xi,\ \zeta \in \mathbb{R}^d : \quad |\mathcal{K}(e, w)\xi{:}\zeta| \leq C_{\mathcal{K}}|\xi||\zeta|\,. \tag{3.10}$$

Data qualification. We shall suppose for the right-hand sides F, H, f, and h that

$$F \in L^2(0, T; W^{1,2}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d)^*) \cap W^{1,1}(0, T; W^{1,p}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d)^*)\,, \tag{3.11a}$$

$$f \in\ L^2(0, T; L^{2(d-1)/d}(\Gamma_{\rm N}; \mathbb{R}^d)) \cap W^{1,1}(0, T; L^1(\Gamma_{\rm N}; \mathbb{R}^d))\,, \tag{3.11b}$$

$$H \in L^1(0, T; L^1(\Omega)), \quad H \geq 0 \text{ a.e. in } Q\,, \tag{3.11c}$$

$$h \in L^1(0, T; L^1(\partial\Omega)), \quad h \geq 0 \text{ a.e. in } (0, T) \times \partial\Omega\,. \tag{3.11d}$$

We also introduce the functions

$$\begin{array}{l} \mathrm{F} : (0, T) \to W^{1,p}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d)^*,\ \langle \mathrm{F}(t), v\rangle := \int_\Omega F(t)v\, \mathrm{d}x + \int_{\Gamma_{\rm N}} f(t)v\, \mathrm{d}S \ \text{ for } v{\in}W^{1,p}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d), \\ \mathrm{H} : (0, T) \to W^{1,r}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d)^*,\ \langle \mathrm{H}(t), v\rangle := \int_\Omega H(t)v\, \mathrm{d}x + \int_{\partial\Omega} h(t)v\, \mathrm{d}S \ \text{ for } v{\in}W^{1,r}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d), \end{array} \tag{3.12}$$

with $1 \leq r < \frac{d+2}{d+1}$, cf. (3.26c) later on. Finally, we impose the following on the initial data

$$u_0 \in W^{1,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d)\,, \quad [\![u_0]\!] \succeq 0 \ \text{ on } (0, T) \times \Gamma_{\rm C}, \tag{3.13a}$$

$$z_0 \in L^\infty(\Gamma_{\rm C}), \qquad 0 \leq z_0 \leq 1 \ \text{ a.e. on}\, \Gamma_{\rm C}\,, \tag{3.13b}$$

$$\theta_0 \in L^{\omega_1}(\Omega)\,, \qquad \theta_0 \geq 0 \ \text{ a.e. in}\, \Omega\,, \tag{3.13c}$$

where ω_1 is the same as in (3.8b). It follows from (3.13c) and (3.8b) that $w_0 := h(\theta_0) \in L^1(\Omega)$.

3.3 General energetic formulation

In the weak formulation for the SBV-brittle delamination system and for its approximations, a crucial role is played by the *mechanical* part of the overall Helmholtz free energy, i.e. by the functional $\Phi : W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \times \mathcal{Z} \to (-\infty,+\infty]$ (with the space $\mathcal{Z}$ specified below), given by $\Phi(u,z) := \Phi^{\rm bulk}(u) + \Phi^{\rm surf}([\![u]\!],z)$, cf. (1.4). In fact, the functional $\Phi^{\rm surf} : L^2(\Gamma_{\rm C}) \times \mathcal{Z} \to (-\infty,+\infty]$ is the only contribution in the mechanical energy Φ to change when passing from the Modica-Mortola, to the SBV-adhesive contact, to the SBV-brittle delamination systems, whereas the bulk contribution $\Phi^{\rm bulk} : W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \to [0,+\infty)$ for all of the three models is given by

$$\Phi^{\rm bulk}(u) := \int_{\Omega\backslash\Gamma_{\rm C}} \big({\rm W}_2(e(u)) + {\rm W}_p(e(u))\big)\,{\rm d}x \quad \text{with} \quad {\rm W}_n(e) := \tfrac{1}{n}|e|^n\,, \ \ n \in \{2,p\}\,. \tag{3.14}$$

In order to specify the surface mechanical energies, we observe that the impenetrability constraint $[\![u]\!]\cdot{\rm n} \geq 0$ on $(0,T)\times\Gamma_{\rm C}$ can be reformulated as

$$[\![u(t,x)]\!] \in {\rm C}(x) \quad \text{for a.a. } (t,x) \in (0,T)\times\Gamma_{\rm C},$$

upon introducing the multivalued mapping

$${\rm C} : \Gamma_{\rm C} \rightrightarrows \mathbb{R}^d \text{ s.t. } {\rm C}(x) = \{v \in \mathbb{R}^d;\ v\cdot{\rm n}(x) \geq 0\} \text{ for a.a. } x \in \Gamma_{\rm C}. \tag{3.15}$$

We will denote by $I_{{\rm C}(x)}$ the indicator functional of the closed cone ${\rm C}(x)$, and by $\partial I_{{\rm C}(x)}$ its (convex analysis) subdifferential.

Then, the surface contributions to the mechanical energy are

– for the *Modica-Mortola adhesive* system:

$$\begin{aligned}\Phi^{\rm surf} = \Phi^{\rm surf}_{k,m}([\![u]\!],z) &:= \int_{\Gamma_{\rm C}} \big(I_{{\rm C}(x)}([\![u]\!]) + J_k([\![u]\!],z) + I_{[0,1]}(z) - a_0 z\big)\,{\rm d}S + \mathcal{G}_m(z) \\ &\qquad\qquad \text{with} \ \ J_k([\![u]\!],z) := \frac{k}{2}z\big|[\![u]\!]\big|^2.\end{aligned} \tag{3.16}$$

We denote by $\Phi_{k,m}$ the corresponding mechanical energy, defined on $W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\times\mathcal{Z}_{\rm MM}$, with

$$\mathcal{Z}_{\rm MM} := H^1(\Gamma_{\rm C}); \tag{3.17}$$

– for the SBV-*adhesive* system:

$$\Phi^{\rm surf} = \Phi^{\rm surf}_{k}([\![u]\!],z) = \int_{\Gamma_{\rm C}} \big(I_{{\rm C}(x)}([\![u]\!]) + J_k([\![u]\!],z) + I_{[0,1]}(z) - a_0 z\big)\ {\rm d}S + \mathcal{G}_{\rm b}(z) \tag{3.18}$$

with

$$\mathcal{G}_{\rm b}(z) = \begin{cases} {\rm b}\mathcal{H}^{d-2}({\rm J}_z) & \text{if } z \in {\rm SBV}(\Gamma_{\rm C};\{0,1\}), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.19}$$

where ${\rm J}_z$ denotes the set of approximate jump points of z (cf. Def. A.13) and, from the calculations for the Γ-limit passage as $m \to \infty$ in the Modica-Mortola functionals $(\mathcal{G}_m)_m$ (see [Mod87, Alb00]), it follows that ${\rm b} = 2\int_0^1 \xi(1-\xi)\,{\rm d}\xi$. We denote by Φ_k the corresponding mechanical energy, defined on the space $W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\times\mathcal{Z}_{\rm SBV}$, with

$$\mathcal{Z}_{\rm SBV} := {\rm SBV}(\Gamma_{\rm C};\{0,1\}); \tag{3.20}$$

– for the SBV-*brittle* system:

$$\Phi^{\rm surf} = \Phi^{\rm surf}_{\rm b}([\![u]\!],z) = \int_{\Gamma_{\rm C}} \big(I_{{\rm C}(x)}([\![u]\!]) + J_\infty([\![u]\!],z) + I_{[0,1]}(z) - a_0 z\big)\ {\rm d}S + \mathcal{G}_{\rm b}(z), \tag{3.21}$$

cf. (2.4) for the definition of J_∞. We denote by $\Phi_{\rm b}$ the corresponding mechanical energy, defined on the space $W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\times\mathcal{Z}_{\rm SBV}$.

As in (1.1) we also introduce the dissipation potential $\mathcal{R}_1 : L^1(\Gamma_C) \times L^1(\Gamma_C) \to [0, +\infty]$

$$\mathcal{R}_1(\tilde{z}-z) := \int_{\Gamma_C} \mathrm{R}_1(\tilde{z}-z) = \begin{cases} \int_{\Gamma_C} a_1|\tilde{z}-z| \, \mathrm{d}S & \text{if } \tilde{z} \le z \text{ a.e. in } \Gamma_C, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.22}$$

In view of the bulk term with p-growth in (3.14) and the surface energies (3.16) and (3.21), we shall use the following notation for sets of test functions for the weak formulation of the momentum equation

$$\mathcal{U} := \Big\{ v \in W^{1,p}_{\Gamma_D}(\Omega\backslash\Gamma_C; \mathbb{R}^d) \; : \; [\![v(x)]\!] \in \mathrm{C}(x) \text{ for a.a. } x \in \Gamma_C \Big\} ; \tag{3.23}$$

$$\mathcal{U}_z := \Big\{ v \in W^{1,p}_{\Gamma_D}(\Omega\backslash\Gamma_C; \mathbb{R}^d) \; : \; [\![v(x)]\!] \in \mathrm{C}(x), \; z(x)[\![v(x)]\!] = 0 \text{ for a.a. } x \in \Gamma_C \Big\} \tag{3.24}$$

with a given $z \in L^1(\Gamma_C)$. The former set is used in the adhesive and the latter in the brittle setting.

For the enthalpy equation, we shall use test functions in the space

$$\mathcal{W} := \mathrm{C}^0([0,T]; W^{1,r'}(\Omega\backslash\Gamma_C)) \cap W^{1,r'}(0,T; L^{r'}(\Omega)) \subset \mathrm{C}^0([0,T]; L^\infty(\Gamma_C)) \tag{3.25}$$

where $r' = \frac{r}{r-1}$ is the conjugate exponent of r in (3.26c) below. Since $1 \le r < \frac{d+2}{d+1}$, by trace embedding (3.2) the inclusion in (3.25) holds. In turn, we may mention that the $L^r(0,T; W^{1,r}(\Omega\backslash\Gamma_C))$-regularity for w derives from BOCCARDO-GALLOUËT-type estimates [BG89] on the enthalpy equation, combined with the Gagliardo-Nirenberg inequality. We refer to the proof of the forthcoming Proposition 3.12, and to [Rou10] for all details.

We are now in the position to introduce a *general* weak solvability notion for a thermal delamination system, i.e. the Modica-Mortola/SBV-adhesive, and SBV-brittle systems, consisting of the weak formulation of the momentum equation, of a mechanical energy inequality, a semistability condition, and of the variational formulation of the enthalpy equation. While the last three items have the same form for each of the delamination systems we consider, we will not give a unified variational formulation of the momentum equation, for it substantially changes when switching from *adhesive* to *brittle* delamination (see Lemma 3.6 later on). In particular, let us highlight that in the brittle case the set of test functions $\mathcal{U}_z$ for the weak formulation of the momentum equation does depend on the z-component of the solution.

Definition 3.3 (Energetic solution) Given a quadruple of initial data $(u_0, \dot{u}_0, z_0, \theta_0)$ satisfying (3.13), we call a triple (u, z, w) an *energetic solution* to the of a thermal delamination system, if

$$u \in L^\infty(0,T; W^{1,p}_{\Gamma_D}(\Omega\backslash\Gamma_C; \mathbb{R}^d)) \cap W^{1,2}(0,T; W^{1,2}_{\Gamma_D}(\Omega\backslash\Gamma_C; \mathbb{R}^d)), \tag{3.26a}$$

$$z \in L^\infty((0,T) \times \Gamma_C) \cap \mathrm{BV}([0,T]; L^1(\Gamma_C)), \qquad z(t,x) \in [0,1] \text{ for a.a. } (t,x) \in (0,T) \times \Gamma_C \,, \tag{3.26b}$$

$$w \in L^r(0,T; W^{1,r}(\Omega\backslash\Gamma_C)) \cap L^\infty(0,T; L^1(\Omega)) \cap \mathrm{BV}([0,T]; W^{1,r'}(\Omega\backslash\Gamma_C)^*) \tag{3.26c}$$

for every $1 \le r < \frac{d+2}{d+1}$, the triple (u, z, w) complies with the initial conditions

$$u(0) = u_0 \quad \text{a.e. in } \Omega, \qquad z(0) = z_0 \quad \text{a.e. in } \Gamma_C, \qquad w(0) = w_0 \quad \text{a.e. in } \Omega. \tag{3.27}$$

and with

1. the weak formulation of the momentum equation
   -in the *adhesive* case:

$$\begin{aligned} & u(t) \in \mathcal{U} \;\; \text{for a.a. } t \in (0,T), \quad \text{ and for all } v \in \mathcal{U} \\ & \int_{\Omega\backslash\Gamma_C} \big(\mathrm{DR}_2(e(\dot{u}(t))) + \mathrm{DW}_2(e(u(t))) - \mathbb{B}\Theta(w(t))) + \mathrm{DW}_p(e(u(t)))\big) : e(v-u(t)) \, \mathrm{d}x \\ & \qquad + \int_{\Gamma_C} kz(t)[\![u(t)]\!] \cdot [\![v-u(t)]\!] \mathrm{d}S \ge \langle \mathrm{F}(t), v-u(t) \rangle \qquad \text{for a.a. } t \in (0,T); \end{aligned} \tag{3.28a}$$

   -in the *brittle* case:

$$\begin{aligned} & u(t) \in \mathcal{U}_{z(t)} \;\; \text{for a.a. } t \in (0,T), \quad \text{ and for all } v \in \mathcal{U}_{z(t)} \\ & \int_{\Omega\backslash\Gamma_C} \big(\mathrm{DR}_2(e(\dot{u}(t))) + \mathrm{DW}_2(e(u(t))) - \mathbb{B}\Theta(w(t))) + \mathrm{DW}_p(e(u(t)))\big) : e(v-u(t)) \, \mathrm{d}x \\ & \qquad \ge \langle \mathrm{F}(t), v-u(t) \rangle \qquad \text{for a.a. } t \in (0,T); \end{aligned} \tag{3.28b}$$

2. semistability for a.a. $t \in (0,T)$

$$\forall \tilde{z} \in \mathcal{Z}: \qquad \Phi\big(u(t), z(t)\big) \leq \Phi\big(u(t), \tilde{z}\big) + \mathcal{R}_1\big(\tilde{z} - z(t)\big); \tag{3.29}$$

3. mechanical energy inequality

$$\begin{aligned} &\Phi\big(u(t), z(t)\big) + \int_0^t 2\,\mathcal{R}_2(e(\dot{u}))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_1}(z;[0,t]) \\ &\quad \leq \Phi\big(u_0, z_0\big) + \int_0^t \int_{\Omega\backslash\Gamma_\mathrm{C}} \Theta(w)\mathbb{B}{:}e(\dot{u})\,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathrm{F}, \dot{u}\rangle\,\mathrm{d}s \qquad \text{for all } t \in [0,T], \end{aligned} \tag{3.30}$$

where we use the notation

$$\mathrm{Var}_{\mathcal{R}_1}(\tilde{z};[t_1,t_2]) := \sup \sum_{i=1}^{k} \mathcal{R}_1\big(\tilde{z}(s_i) - \tilde{z}(s_{i-1})\big) \ \text{ for } \ \tilde{z} \in L^1(\Gamma_\mathrm{C}), \ [t_1,t_2] \subset [0,T], \tag{3.31}$$

with the sup taken over all partitions $t_1 = s_0 < \ldots < s_k = t_2$ of the interval $[t_1,t_2]$;

4. weak enthalpy inequality

$$\begin{aligned} &\langle w(T), \zeta(T)\rangle + \int_0^T \int_{\Omega\backslash\Gamma_\mathrm{C}} \mathcal{K}(e(u),w)\nabla w{\cdot}\nabla\zeta - w\dot{\zeta}\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_\mathrm{C}} \eta(x, [\![u]\!], z)[\![\Theta(w)]\!]\,[\![\zeta]\!]\,\mathrm{d}S\mathrm{d}t \\ &\geq \int_0^T \int_{\Omega\backslash\Gamma_\mathrm{C}} \big(2\mathrm{R}_2(e(\dot{u}))|\zeta| - \Theta(w)\mathbb{B}\,{:}e(\dot{u})\zeta\big)\,\mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\Gamma_\mathrm{C}} \frac{\zeta|^+_{\Gamma_\mathrm{C}} + \zeta|^-_{\Gamma_\mathrm{C}}}{2}\,\mathrm{d}\xi^{\mathrm{surf}}_{\dot{z}}(S,t) \\ &\quad + \int_0^T \langle \mathrm{H}, \zeta\rangle \mathrm{d}t + \int_{\Omega\backslash\Gamma_\mathrm{C}} w_0\zeta(0)\,\mathrm{d}x \quad \text{for all } \zeta \in \mathcal{W}, \end{aligned} \tag{3.32}$$

where $w_0 = h(\theta_0)$ and $\xi^{\mathrm{surf}}_{\dot{z}}$ is a measure (=heat produced by rate-independent dissipation) defined by prescribing its values for every closed set of the type $A := [t_1,t_2]{\times}C \subset [0,T] \times \overline{\Gamma_\mathrm{C}}$ by

$$\xi^{\mathrm{surf}}_{\dot{z}}(A) := \int_C \mathrm{R}_1(z(t_1,x) - z(t_2,x))\,\mathrm{d}S\,. \tag{3.33}$$

Notice that, since $w \in \mathrm{BV}([0,T]; W^{1,r'}(\Omega\backslash\Gamma_\mathrm{C})^*)$, for all $t \in [0,T]$ one has $w(t) \in W^{1,r'}(\Omega\backslash\Gamma_\mathrm{C})^*$, so that the first duality pairing on the left-hand side of (3.32) makes sense pointwise.

Remark 3.4 (Consistency with the energetic solutions in the rate-independent case) Note that, without viscosity in the momentum equation and in the isothermal case (i.e., in the case of a purely *rate-independent* evolution of delamination, cf. [RSZ09]), the notion of weak solution of Definition 3.3 coincides with the concept of (global) energetic solution introduced in [MT04], see also [Mie05].

Remark 3.5 (Total energy inequality) Adding the mechanical energy inequality (3.30) (for $t = T$), and the weak formulation (3.32) of the enthalpy equation tested by 1 yields a further energy inequality

$$\Phi\big(u(T), z(T)\big) + \int_0^T 2\,\mathcal{R}_2(e(\dot{u}))\,\mathrm{d}s + \langle w(T), 1\rangle \leq \Phi\big(u_0, z_0\big) + \int_{\Omega\backslash\Gamma_\mathrm{C}} w_0\,\mathrm{d}x + \int_0^T \langle \mathrm{F}, \dot{u}\rangle\,\mathrm{d}t + \int_0^T \langle \mathrm{H}, 1\rangle\,\mathrm{d}t, \tag{3.34}$$

which involves the enthalpy contribution $\langle w(T), 1\rangle$ as well.

However, for the adhesive systems it is possible to prove even equalities in the energy inequalities (3.30), (3.34) and in the enthalpy inequality (3.32). This is related to obtaining convergence of the quadratic dissipation term on the right-hand side of (3.32).

Theorem 3.1 (Energy and enthalpy equalities for the adhesive systems) *Assume* (3.7)*,* (3.8)*,* (3.11)*, and* (3.13)*. Let the energy of the adhesive contact system be given by either $\Phi_{k,m}$ from* (3.16) *for any $m, k > 0$ fixed, or by Φ_k from* (3.18) *for any $k > 0$ fixed. Then the mechanical energy inequality* (3.30)*, the enthalpy inequality* (3.32) *as well as the total energy inequality* (3.34) *hold as equalities.*

The proof will be given in Section 4.3 for SBV-adhesive contact, i.e. for the energy Φ_k; for Modica-Mortola adhesive contact one uses exactly the same arguments. They amount to first showing the opposite relation in the mechanical energy estimate (3.30) by means of Riemann-sum arguments (developed in Sec. 4.3) on the momentum balance and the semistability inequality. This yields the mechanical energy equality, which then allows us to deduce convergence of the viscous dissipation, crucial to obtain the enthalpy *equality*. Finally, summing the mechanical and enthalpy equalities leads to the total energy equality. Let us mention here that the analog of Thm. 3.1 cannot be obtained in the brittle setting, where already the strategy to gain the mechanical energy balance fails. The reasons for this are discussed in Remark 4.5.

In fact, in order to obtain the opposite relation in the mechanical energy inequality for the adhesive models, we will not employ the momentum balance as a variational inequality but consider its reformulation as a subdifferential inclusion, as stated in the following

Lemma 3.6 (Subdifferential formulation of the momentum equation)
Assume (3.7).

1. For $I_{\rm C}$ from (3.15) and J_k from (3.16) consider the functionals

$$\mathcal{I}_{\rm C} : W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \to [0,+\infty], \quad \mathcal{I}_{\rm C}(u) = \int_{\Gamma_{\rm C}} I_{{\rm C}(x)}(\llbracket u(x) \rrbracket)\,{\rm d}S, \tag{3.35}$$

$$\mathcal{J}_k : W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\times L^\infty(\Gamma_{\rm C}) \to [0,+\infty], \quad \mathcal{J}_k(u,z) = \int_{\Gamma_{\rm C}} J_k(\llbracket u \rrbracket, z)\,{\rm d}S = \tfrac{k}{2}\int_{\Gamma_{\rm C}} z|\llbracket u \rrbracket|^2\,{\rm d}S, \tag{3.36}$$

$$\mathcal{F}_k(u,z) := \mathcal{I}_{\rm C}(u) + \mathcal{J}_k(u,z), \tag{3.37}$$

with subdifferentials $\partial\,\mathcal{I}_{\rm C}$, $\partial_u\mathcal{J}_k$, $\partial_u\mathcal{F}_k : W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \rightrightarrows W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*$ (∂_u denoting the subdifferential w.r.t. u). Then, the *sum rule*

$$\partial_u\mathcal{F}_k(u,z) = \partial\,\mathcal{I}_{\rm C}(u) + \partial_u\mathcal{J}_k(u,z) \qquad \text{holds for all } (u,z)\in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\times L^\infty(\Gamma_{\rm C}), \ \ \text{viz.}$$
$$\lambda \in \partial_u\mathcal{F}_k(u,z) \ \Leftrightarrow\ \exists\,\ell\in\partial\,\mathcal{I}_{\rm C}(u) \text{ s.t. } \forall\, v\in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\ \ \langle\lambda,v\rangle = \langle\ell,v\rangle + \int_{\Gamma_{\rm C}} kz\llbracket u\rrbracket\cdot\llbracket v\rrbracket\,{\rm d}S, \tag{3.38}$$

and (3.28a) is equivalent to

for all $v\in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$, for a.a. $t\in(0,T)$:

$$\int_{\Omega\backslash\Gamma_{\rm C}}\big({\rm DR}_2(e(\dot u(t)))+{\rm DW}_2(e(u(t)))-\mathbb{B}\Theta(w(t))+{\rm DW}_p(e(u(t)))\big){:}e(v)\,{\rm d}x$$
$$+\underbrace{\int_{\Gamma_{\rm C}} kz(t)\llbracket u(t)\rrbracket\cdot\llbracket v\rrbracket\,{\rm d}S + \langle\ell(t),v\rangle}_{\langle\lambda(t),v\rangle} = \langle {\rm F}(t),v\rangle \tag{3.39}$$

with $\ell\in L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*)$ such that $\ell(t)\in\partial\mathcal{I}_{\rm C}(u(t))$ for a.a. $t\in(0,T)$
and $\lambda\in L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*)$ such that $\lambda(t)\in\partial_u\mathcal{F}_k(u(t),z(t))$ for a.a. $t\in(0,T)$,

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p.

2. For $I_{\rm C}$ from (3.15) and J_∞ from (3.21) consider the functionals

$$\mathcal{J}_\infty : W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)\times L^\infty(\Gamma_{\rm C}) \to [0,+\infty], \quad \mathcal{J}_\infty(u,z) := \int_{\Gamma_{\rm C}} J_\infty\big(\llbracket u(x)\rrbracket, z(x)\big)\,{\rm d}S, \tag{3.40}$$

$$\mathcal{F}_\infty(u,z) := \mathcal{I}_{\rm C}(u) + \mathcal{J}_\infty(u,z). \tag{3.41}$$

Then, (3.28b) can be reformulated as

for all $v\in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ and a.a. $t\in(0,T)$:

$$\int_{\Omega\backslash\Gamma_{\rm C}}\big({\rm DR}_2(e(\dot u(t)))+{\rm DW}_2(e(u(t)))-\mathbb{B}\Theta(w(t))+{\rm DW}_p(e(u(t)))\big){:}e(v)\,{\rm d}x + \langle\lambda(t),v\rangle = \langle {\rm F}(t),v\rangle \tag{3.42}$$

with $\lambda\in L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*)$ such that $\lambda(t)\in\partial_u\mathcal{F}_\infty(u(t),z(t))$ for a.a. $t\in(0,T)$.

Observe that the sum rule (3.38) holds thanks to the Rockafellar-Moreau theorem, see e.g. [IT79, p. 200, Thm. 1]. For $\mathcal{F}_\infty$ we only have $\partial\mathcal{I}_{\mathrm{C}} + \partial_u \mathcal{J}_\infty \subset \partial_u \mathcal{F}_\infty$, whereas the converse inclusion in fact may not hold.

Now, we specialize Definition 3.3 to the delamination systems considered in what follows.

Definition 3.7 (Energetic solution of the *Modica-Mortola adhesive* contact system) Given a quadruple of initial data $(u_0, \dot{u}_0, z_0, \theta_0)$ satisfying (3.13), we call a triple (u, z, w) an *energetic solution* to the *Modica-Mortola adhesive* contact system, if, in addition to (3.26b), we have

$$z \in L^\infty(0,T;\mathcal{Z}_{\mathrm{MM}}) \tag{3.43}$$

with $\mathcal{Z}_{\mathrm{MM}}$ from (3.17), the triple (u, z, w) fulfills Definition 3.3, with the weak formulation of the momentum inclusion (3.28a), and Φ replaced by $\Phi_{k,m}$ from (3.16).

Definition 3.8 (Energetic solution of the SBV-*adhesive* contact system) Given a quadruple of initial data $(u_0, \dot{u}_0, z_0, \theta_0)$ satisfying (3.13), we call a triple (u, z, w) an *energetic solution* to the SBV-*adhesive* contact system, if, in addition to (3.26b), we have

$$z \in L^\infty(0,T;\mathcal{Z}_{\mathrm{SBV}}) \tag{3.44}$$

with $\mathcal{Z}_{\mathrm{SBV}}$ from (3.20), the triple (u, z, w) fulfills Definition 3.3, with the weak formulation of the momentum inclusion (3.28a), and Φ replaced by Φ_k from (3.18).

Definition 3.9 (Energetic solution of the SBV-*brittle* delamination system) Given a quadruple of initial data $(u_0, \dot{u}_0, z_0, \theta_0)$ satisfying (3.13), we call a triple (u, z, w) an *energetic solution* to the (Cauchy problem for the) SBV-*brittle* contact system, if (3.44) holds, the triple (u, z, w) fulfills Definition 3.3, with the weak formulation of the momentum inclusion (3.28b), and Φ replaced by Φ_{b} from (3.21).

As already mentioned, the energy and enthalpy equalities of Thm. 3.1 seem to be out of reach for the SBV-*brittle* delamination system. In what follows we discuss a possible integration of the weak formulation (3.32) of the enthalpy equation, by means of the concept of *defect measure*.

Remark 3.10 (*Defect measure* of the enthalpy equation in the brittle case) In our approach, the failure of equality in the weak formulation (3.32) of the enthalpy equation is due to a lack of strong compactness in $L^2(0,T;L^2(\Omega;\mathbb{R}^{d\times d}))$ for the sequence $(e(\dot{u}_k))_k$, where $(u_k, z_k, w_k)_k$ is a sequence of solutions to the SBV-*adhesive* contact problems with which we approximate as $k \to \infty$ the SBV-*brittle* delamination system.

Therefore, the passage to the limit as $k \to \infty$ in the first term on the right-hand side of the enthalpy *equalities* (by Thm. 3.1) for the SBV-*adhesive* contact systems, solely relies on lower semicontinuity arguments, cf. the proof of Thm. 5.1.

Nonetheless, one can consider the limit in the sense of measures of the sequence $(2\mathrm{R}_2(e(\dot{u}_k)))_k$: it is a Radon measure μ_0 on $[0,T] \times \overline{\Omega}$. Taking the limit of (3.32) as $k \to \infty$ then leads to

$$\begin{aligned}
&\langle w(T), \zeta(T)\rangle + \int_0^T \int_\Omega \mathcal{K}(e(u), w)\nabla w\cdot\nabla\zeta - w\dot{\zeta}\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Gamma_{\mathrm{C}}} \eta(x, [\![u]\!], z)[\![\Theta(w)]\!]\,[\![\zeta]\!]\,\mathrm{d}S\mathrm{d}t \\
&= \int_0^T \int_\Omega \big(2\mathrm{R}_2(e(\dot{u})) - \Theta(w)\mathbb{B} : e(\dot{u})\big)\zeta\,\mathrm{d}x\mathrm{d}t + \iint_{(0,T)\times\Gamma_{\mathrm{C}}} \frac{\zeta|^+_{\Gamma_{\mathrm{C}}} + \zeta|^-_{\Gamma_{\mathrm{C}}}}{2}\,\mathrm{d}\xi^{\mathrm{surf}}_{\dot{z}}(S,t) \\
&\quad + \iint_{(0,T)\times\Omega} \zeta\,\mathrm{d}\mu + \int_0^T \langle \mathrm{H}, \zeta\rangle \mathrm{d}t + \int_{\Omega\backslash\Gamma_{\mathrm{C}}} w_0\zeta(0)\,\mathrm{d}x \quad \text{for all } \zeta \in \mathcal{W},
\end{aligned} \tag{3.45}$$

where the measure μ is given by

$$\mu = \mu_0 - 2\mathrm{R}_2(e(\dot{u}))\,\mathrm{d}\mathcal{L}, \tag{3.46}$$

with $\mathrm{d}\mathcal{L}$ is the Lebesgue measure on $(0,T)\times\Omega$. Following [Nau06] (see also [Fei04]), we refer to μ as a *defect measure*, for it represents the defect between the limiting measure μ_0 and the dissipation $2\mathrm{R}_2(e(\dot{u}))$. Hence, as in [Nau06] in the brittle case we could complete the weak enthalpy inequality by coupling it with (3.45)–(3.46).

3.4 Strategy of the existence proof and uniform a priori estimates

Here, we provide the general scheme for proving the existence of solutions to (the Cauchy problems for) the *Modica-Mortola*, SBV-*adhesive*, and SBV-*brittle* delamination systems, when taking the limit in a suitable approximate problem: i.e., passing to the limit either with a time-discretization scheme to the Modica-Mortola system, with the Modica-Mortola system to the SBV-adhesive system, with the SBV-adhesive system to the SBV-brittle system. We will refer to the latter limit passage as the *brittle limit*, and to the former two passages as the *adhesive limit(s)*.

Notation 3.11 Hereafter, we shall suppose that the parameters m and k vary in $\mathbb{N}$. This will allow us to directly consider *sequences* $(u_m, z_m, w_m)_m$ of solutions to the Modica-Mortola delamination system (where we omit the dependence on k for notational simplicity), when taking the limit as $m \to \infty$; *sequences* $(u_k, z_k, w_k)_k$ of solutions to the SBV-adhesive contact system, when taking the limit as $k \to \infty$.

In performing the aforementioned passages to the limit, we shall always follow these steps:

Step 0: *a priori estimates* and *compactness* for the approximate solutions;

Step 1: proof of the weak formulation of the *momentum equation*. To this aim, we shall rely on the subdifferential reformulations of Lemma 3.6, and in all of the adhesive limits, use techniques from maximal monotone operator theory to identify the weak limits of the nonlinear terms. For the brittle limit, we will need to prove MOSCO-convergence as $k \to \infty$ of the functionals $(\mathcal{J}_k)_k$ to the functional $\mathcal{J}_\infty$. Combining this information with maximal monotone operator techniques, we will handle the passage to the limit in the term $\frac{k}{2} z |[\![u]\!]|^2$ as $k \to \infty$.

Step 2: proof of the *semistability* condition (3.29), verifying the *mutual recovery sequence* condition from [MRS08], in Propositions 4.2 and 5.7;

Step 3: proof of the *mechanical energy inequality* (3.30) by lower semicontinuity arguments;

Step 4: proof of the weak formulation of the *enthalpy inequality*.

A priori estimates. We conclude this section by collecting the a priori estimates on approximate solutions, which are valid for all of the successive approximations of the SBV-brittle system we shall tackle: viz., the Modica-Mortola approximation of the SBV-adhesive system; the SBV-adhesive approximation of the SBV-brittle system. In order to state such estimates in a unified way, we consider a generic sequence $(u_n, z_n, w_n)_n$ of *energetic solutions* to the thermal delamination system driven by a sequence $(\Phi_n)_n$ of energy functionals $\Phi_n : W^{1,p}(\Omega \backslash \Gamma_{\mathrm{C}}; \mathbb{R}^d) \times \mathcal{Z} \to (-\infty, +\infty]$. More specifically, when considering

(a1) the Modica-Mortola approximation of the SBV-adhesive system, we have the energies $(\Phi_{k,m})_m$, and $\mathcal{Z} = \mathcal{Z}_{\mathrm{MM}}$: we shall consider the energetic solutions $(u_m, z_m, w_m)_m$ (for notational simplicity, we omit their dependence on $k \in \mathbb{N}$), obtained by passing to the limit in the time-discretization scheme of Problem A.1 in Appendix A.1;

(a2) the SBV-adhesive approximation of the SBV-brittle system, we have the energies $(\Phi_k)_k$, and $\mathcal{Z} = \mathcal{Z}_{\mathrm{SBV}}$: we shall consider the energetic solutions $(u_k, z_k, w_k)_k$ obtained by passing to the limit in the Modica-Mortola approximation, cf. Sec. 4.

We shall call an energetic solution to the Modica-Mortola adhesive (to the adhesive SBV, resp.) delamination system *approximable*, if it is obtained by passing to the limit in the time-discretization scheme of Problem A.1 (in the Modica-Mortola approximation, resp.). We can now state the following general result yielding a priori estimates on the family $(u_n, z_n, w_n)_n$.

Proposition 3.12 (A priori estimates) *Assume* (3.7), (3.8), (3.11), *and let* (u_0, θ_0, z_0) *be a triple of initial data complying with* (3.13). *Suppose in addition that* (u_0, z_0) *comply with the semistability* (3.29) *with the energy* Φ_n, *i.e.*

$$\Phi_n(u_0, z_0) \leq \Phi_n(u_0, \tilde{z}) + \mathcal{R}_1(\tilde{z} - z_0) \qquad \text{for all } \tilde{z} \in \mathcal{Z}.$$

Let $(u_n, z_n, w_n)_n$ *be a family of* energetic solutions *to the thermal delamination system in the* adhesive *case (viz. with* (3.28a)*), in either of the two cases* (a1) *and* (a2).

Then, there exist a constant $S > 0$ and, for every $1 \le r < \frac{d+2}{d+1}$, $S_r > 0$, such that for all $n \in \mathbb{N}$ the following estimates hold:

$$\|u_n\|_{L^\infty(0,T;W^{1,p}_{\Gamma_D}(\Omega;\mathbb{R}^d))\cap W^{1,2}(0,T;W^{1,2}_{\Gamma_D}(\Omega;\mathbb{R}^d))} \le S; \tag{3.47}$$

$$\sup_{t\in[0,T]} \Phi_n\left(u_n(t), z_n(t)\right) \le S; \tag{3.48}$$

$$\|z_n\|_{L^\infty((0,T)\times\Gamma_C)} \le S; \tag{3.49}$$

$$\|z_n\|_{\mathrm{BV}([0,T];L^1(\Gamma_C))} \le S; \tag{3.50}$$

$$\|w_n\|_{L^\infty(0,T;L^1(\Omega))} \le S; \tag{3.51}$$

$$\|w_n\|_{L^r(0,T;W^{1,r}(\Omega))} + \|w_n\|_{\mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)} \le S_r \quad \text{for any } 1 \le r < \tfrac{d+2}{d+1}\,. \tag{3.52}$$

The *proof* follows by lower semicontinuity arguments. Indeed, we start from the time-discretization of the Modica-Mortola system, see Sec. A.1 ahead. For the related approximate solutions, the estimates of Prop. A.6 hold, with a constant independent of the time-step τ, and of the parameters m and k. In view of the convergences (A.15)–(A.23) of the approximate solutions, such estimates are inherited by the *approximable* energetic solutions of the Modica-Mortola delamination system. This yields the bounds (3.47)–(3.52), with a constant independent of m and k. Then, the convergences stated in Thm. 4.3 and again lower semicontinuity arguments ensure that (3.47)–(3.52) are also valid for the *approximable* energetic solutions of the SBV-adhesive system, uniformly w.r.t. the parameter k.

Remark 3.13 (Extension: more general bulk energies) The bulk energy densities $\mathrm{W}_2(e) = \frac{1}{2}e{:}\,\mathbb{C}{:}\,e$ and $\mathrm{W}_p(e) = \frac{1}{p}|e|^{p-2}e{:}\,\mathbb{H}{:}\,e$ can be replaced by general *strictly convex*, Gâteaux-differentiable functions $\mathrm{W}_n : \mathbb{R}^d \to \mathbb{R}$ fulfilling suitable growth assumptions from above and below.

4 Adhesive contact: From Modica-Mortola- to SBV-regularization

The main goal of this section is to prove the existence of energetic solutions in the sense of Definition 3.8 for the SBV-adhesive contact model, and precisely the following

Theorem 4.1 (Existence result for SBV-adhesive contact, $k > 0$ fixed)
Keep $k > 0$ fixed. Assume (3.7), (3.8), (3.11), (3.13). *Suppose that the initial data* (u_0, z_0) *fulfill*

$$\Phi_k(u_0, z_0) \le \Phi_k(u_0, \tilde z) + \mathcal{R}_1(\tilde z - z_0) \qquad \text{for all } \tilde z \in \mathcal{Z}_{\mathrm{SBV}}. \tag{4.1}$$

Then, there exists an energetic solution (u, w, z) *to the* SBV-*adhesive contact system, such that* (u, z) *comply with the semistability* (3.29) for all $t \in [0, T]$. *Moreover, the mechanical energy, the enthalpy and the total energy estimates* (3.30), (3.32) *and* (3.34) *with* Φ_k *hold as equalities. Furthermore,*

$$\exists\, \theta^* > 0 : \inf_{x\in\Omega} \theta_0(x) \ge \theta^* \quad \Rightarrow \quad \exists\, \bar\theta > 0 : \inf_{(t,x)\in(0,T)\times\Omega} \theta(t, x) = \inf_{(t,x)\in(0,T)\times\Omega} \Theta(w(t, x)) \ge \bar\theta. \tag{4.2}$$

To prove this, we apply the following strategy:

1. we start from an existence result for Modica-Mortola adhesive contact, $m, k>0$ fixed (Thm. 4.2),
2. as $m\to\infty$, $k>0$ fixed, we show that the energetic solutions of the Modica-Mortola adhesive contact models suitably converge to an energetic solution of the SBV-adhesive contact model (Thm. 4.3),
3. from Thm. 3.1 we directly conclude the validity of the mechanical, the enthalpy and the total energy balance as equalities.

Indeed, we have

Theorem 4.2 (Existence for the Modica-Mortola adhesive contact model, m, $k > 0$ fixed)
Keep m, $k > 0$ fixed. Assume (3.7), (3.8), (3.11), (3.13). *Suppose that the initial data* (u_0, z_0) *fulfill*

$$\Phi_{k,m}(u_0, z_0) \le \Phi_{k,m}(u_0, \tilde z) + \mathcal{R}_1(\tilde z - z_0) \qquad \text{for all } \tilde z \in \mathcal{Z}_{\mathrm{MM}}. \tag{4.3}$$

Then, there exists an energetic solution (u, w, z) *to the Modica-Mortola adhesive contact system, such that* (u, z) *complies with semistability* (3.29) for all $t\in[0, T]$. *Moreover, the energy estimates* (3.30), (3.32) *and* (3.34) *with* $\Phi_{k,m}$ *hold as equalities. Furthermore,* (4.2) *holds.*

The proof of Thm. 4.2 follows from passing to the limit in a suitably devised semi-implicit time-discretization scheme, which we detail in Appendix A.1. Therein, we will also sketch the main steps of the passage to the limit in the time-discretization, and specifically dwell on the differences between our argument and the arguments in [RR11b, RR11a], where a semi-implicit discretization procedure was also developed for proving existence to adhesive contact models in thermo-visco-elasticity. In particular, we will detail the proof of the semistability condition (3.29), which needs to be carefully handled due to the *gradient regularization* in the subdifferential inclusion (2.14) for z.

Concerning the convergence of the Modica-Mortola approximation to SBV-adhesive contact, we have

Theorem 4.3 (Modica-Mortola approximation of SBV-adhesive contact, $k > 0$ fixed) *Keep $k > 0$ fixed. Assume (3.7), (3.8), (3.11). Let $(u_m, w_m, z_m)_m$ be a sequence of* approximable *solutions to the Modica-Mortola adhesive model, supplemented with initial data $(u_m^0, \theta_m^0, z_m^0)_m$ fulfilling (3.13) and (4.3). Suppose that, as $m \to \infty$*

$$u_m^0 \rightharpoonup u_0 \ \text{ in } W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d), \quad \theta_m^0 \to \theta_0 \ \text{ in } L^{\omega_1}(\Omega), \quad z_m^0 \overset{*}{\rightharpoonup} z_0 \ \text{ in } L^\infty(\Gamma_{\rm C}), \text{ and} \tag{4.4}$$

$$\Phi_{k,m}(u_m^0, z_m^0) \to \Phi_k(u_0, z_0). \tag{4.5}$$

Then, there exist a (not relabeled) subsequence, and a triple (u, w, z), such that the following convergences hold as $m \to \infty$

$$u_m \rightharpoonup u \qquad \text{in } L^\infty(0,T;W^{1,p}_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d)) \cap W^{1,2}(0,T;W^{1,2}_{\Gamma_{\rm D}}(\Omega;\mathbb{R}^d)), \tag{4.6a}$$

$$u_m \to u \qquad \text{in } \mathrm{C}^0([0,T];W^{1-\epsilon,p}_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)) \text{ for all } \epsilon \in (0,1], \tag{4.6b}$$

$$z_m \overset{*}{\rightharpoonup} z \qquad \text{in } L^\infty(0,T;\mathrm{SBV}(\Gamma_{\rm C};\{0,1\})) \cap L^\infty((0,T)) \times \Gamma_{\rm C}), \tag{4.6c}$$

$$z_m(t) \overset{*}{\rightharpoonup} z(t) \qquad \text{in } \mathrm{SBV}(\Gamma_{\rm C};\{0,1\}) \cap L^\infty(\Gamma_{\rm C}), \tag{4.6d}$$

$$z_m(t) \to z(t) \qquad \text{in } L^q(\Gamma_{\rm C}) \text{ for all } 1 \le q < \infty \text{ for all } t \in [0,T], \tag{4.6e}$$

$$z_m \to z \qquad \text{in } L^q(0,T;L^q(\Gamma_{\rm C})) \text{ for all } 1 \le q < \infty, \tag{4.6f}$$

$$w_m \rightharpoonup w \qquad \text{in } L^r(0,T;W^{1,r}(\Omega\backslash\Gamma_{\rm C})), \tag{4.6g}$$

$$w_m \to w \qquad \text{in } L^r(0,T;W^{1-\epsilon,r}(\Omega\backslash\Gamma_{\rm C})) \cap L^q(0,T;L^1(\Omega)) \text{ for all } \epsilon \in (0,1],\ 1 \le q < \infty, \tag{4.6h}$$

$$w_m(t) \rightharpoonup w(t) \qquad \text{in } W^{1,r'}(\Omega\backslash\Gamma_{\rm C})^* \text{ for all } t \in [0,T], \tag{4.6i}$$

$$\Theta(w_m) \to \Theta(w) \qquad \text{in } L^2(0,T;L^2(\Omega)), \tag{4.6j}$$

$$[\![\Theta(w_m)]\!] \to [\![\Theta(w)]\!] \qquad \text{in } L^{r\omega}(0,T;L^{(s-\epsilon)\omega}(\Gamma_{\rm C})) \text{ for all } 0 < \epsilon \le s-1, \tag{4.6k}$$

and (u, w, z) is an energetic solution *to the SBV-adhesive contact system. Furthermore, (4.2) holds for $\theta = \Theta(w)$.*

Proof: In order to develop the proof of the passage to the limit in the Modica-Mortola approximation, we now follow the steps outlined in Sec. 3.4.

Step 0: selection of converging subsequences. Estimates (3.47)–(3.52) hold for the sequence $(u_m, w_m, z_m)_m$. Convergences (4.6a), (4.6b) follow from standard weak and strong compactness results (cf. the Aubin-Lions type theorems in [Sim87, Cor. 4, Cor. 5]). Taking into account that $p > d \ge 2$, Sobolev trace theorems (cf. (3.2)) and embedding results, from (4.6b) we deduce that

$$[\![u_m]\!] \to [\![u]\!] \quad \text{in } \mathrm{C}^0([0,T];\mathrm{C}^0(\Gamma_{\rm C};\mathbb{R}^d)). \tag{4.7}$$

As for $(z_m)_m$, the L^∞-convergence in (4.6c) ensues from (3.49) via the Banach selection principle. To obtain the weak*-SBV convergences in (4.6c) and (4.6d), we exploit estimate (3.48), which implies that $\mathcal{G}_m(z_m(t)) \le C$ for a constant independent of m and t. Therefore, in view of the well-known compactness and Γ-convergence result for the static Modica-Mortola functional recalled in Theorem 4.4 below, the sequence $(z_m(t))_m$ is precompact in $L^1(\Gamma_{\rm C})$. The strong L^q-convergence for any $q \in [1,\infty)$, see (4.6e) and (4.6f), is then implied by the uniform L^∞-bound (3.49). From this we directly conclude

$$\mathcal{R}_1(z(s){-}z(t)) = \mathrm{Var}_{\mathcal{R}_1}(z;[s,t]) = \lim_{m\to\infty} \mathrm{Var}_{\mathcal{R}_1}(z_m;[s,t]) = \lim_{m\to\infty} \mathcal{R}_1(z_m(s){-}z_m(t)) \tag{4.8}$$

for all $0 \leq s \leq t \leq T$. From (4.13) below, we also deduce that $\liminf_{m\to\infty} \mathcal{G}_m(z_m(t)) \geq \mathcal{G}_{\mathrm{b}}(z(t))$ for all $t \in [0,T]$. Then, taking into account (4.6a), (4.6d), (4.6e), and (4.7), we have

$$\liminf_{m\to\infty} \Phi_{k,m}(u_m(t), z_m(t)) \geq \Phi_k(u(t), z(t)) \qquad \text{for all } t \in [0,T]. \tag{4.9}$$

As for $(w_m)_m$, convergences (4.6g)–(4.6h) are a consequence of estimates (3.51)–(3.52), and of a generalization of the Aubin-Lions theorem to the case of time derivatives as measures (see e.g. [Rou05, Cor. 7.9]). Taking into account the a priori bound of $(w_m(t))_m$ in $L^1(\Omega)$, we then conclude (4.6i). Furthermore, arguing by interpolation (e.g. via the Gagliardo-Nirenberg inequality), it is possible to derive from (4.6h) that

$$w_m \to w \qquad \text{in } L^{(d+2)/d-\epsilon}(0,T; L^{(d+2)/d-\epsilon}(\Omega)) \qquad \text{for all } 0 < \epsilon \leq \frac{d+2}{d} - 1, \tag{4.10}$$

see [Rou10, Sec. 4] for further details. Hence, relying on the growth condition (3.9) for Θ and on the fact that $\omega > \frac{2d}{d+2}$, one can tune $\epsilon > 0$ in (4.10) in such a way as to obtain (4.6j). Moreover, again taking into account the trace result (3.2), we deduce from (4.6h) that, $w_m^i|_{\Gamma_{\mathrm{C}}} \to w^i|_{\Gamma_{\mathrm{C}}}$ in $L^r(0,T; L^{s-\epsilon}(\Gamma_{\mathrm{C}}))$ for all $0 < \epsilon \leq s-1$ with $s = \frac{(d-1)r}{d-r}$, for $i = +, -$. Therefore, (3.9) ensures (4.6k).

Steps 1 **and** 2, i.e. the limit passages in the **momentum balance** and in the **semistability condition** will be carried out separately in Subsections 4.1 and 4.2, respectively.

Step 3: **mechanical energy inequality.** We use (4.6a), (4.8), and (4.9) to pass to the limit as $m \to \infty$ on the left-hand side of the mechanical energy inequality (3.30) for the Modica-Mortola solutions $(u_m, w_m, z_m)_m$. Combining (4.6a) and (4.6j), we have

$$\Theta(w_m)\mathbb{B}{:}e(\dot{u}_m) \rightharpoonup \Theta(w)\mathbb{B}{:}e(\dot{u}) \qquad \text{in } L^1(0,T; L^1(\Omega)). \tag{4.11}$$

This, (4.5), and again (4.6a) enable us to pass to the limit on the right-hand side of (3.30), and thus to conclude that (u, w, z) complies with the mechanical energy inequality for the SBV-adhesive system.

Step 4: **enthalpy inequality.** Thanks to convergence (4.6i) we pass to the limit as $m \to \infty$ in the first term on the left-hand side of (3.32). We deal with the second integral by means of (4.6g), which we combine with the convergence $\mathcal{K}(e(u_m), w_m) \to \mathcal{K}(e(u), w)$ in $L^q(0,T; L^q(\Omega))$ for all $1 \leq q < \infty$ via (4.6b), (4.6h), and the boundedness of $\mathcal{K}$. To pass to the limit in the surface integral term on the left-hand side, we rely on (4.6k) and on (4.6f) and (4.7), which yield $\eta([\![u_m]\!], z_m) \to \eta([\![u]\!], z)$ in $L^q(0,T; L^q(\Gamma_{\mathrm{C}}))$ for all $1 \leq q < \infty$ in view of the at most polynomial growth (3.8e) of η. The passage to the limit in the first term on the right-hand side of (3.32) is guaranteed by (4.6a) via lower semicontinuity, and by (4.11). For the second term, we observe that

$$\xi^{\mathrm{surf}}_{\dot{z}_m} \to \xi^{\mathrm{surf}}_{\dot{z}} \qquad \text{in the sense of measures on } (0,T) \times \Gamma_{\mathrm{C}}.$$

This follows from the fact that $\mathrm{Var}_{\mathcal{R}_1}(z_m, [0,T]) \to \mathrm{Var}_{\mathcal{R}_1}(z, [0,T])$ (cf. (4.8)), arguing in the very same way as in [Rou10, Sec. 4]. Finally, observe that the strong convergence $\theta_m^0 \to \theta_0$ in $L^{\omega_1}(\Omega)$ and the growth condition (3.8b) yield that $w_m^0 := h(\theta_m^0) \to w_0 := h(\theta_0)$ in $L^1(\Omega)$, which allows us to take the limit of the last term on the right-hand side. Thus, the triple (u, w, z) fulfills the enthalpy inequality (3.32).

Positivity of the temperature. Suppose that $\inf_{x\in\Omega} \theta_0(x) \geq \theta^* > 0$: it follows from convergence (4.4) that there exist $\bar{m} \in \mathbb{N}$ and $\tilde{\theta} > 0$ such that $\inf_{x\in\Omega} \theta_m^0(x) \geq \tilde{\theta}$ for all $m \geq \bar{m}$. Then, by Thm. 4.2 (cf. also (A.29) later on) there exists $\bar{\theta} > 0$ with $\inf_{(t,x)\in(0,T)\times\Omega} \theta_m(t,x) \geq \bar{\theta}$ for all $m \geq \bar{m}$, and (4.2) ensues from convergence (4.6j). This concludes the proof of Theorem 4.3. ∎

For the convergence results (4.6e), (4.6f) we exploited the well-known Γ-convergence theorem for the *static* functionals $(\mathcal{G}_m)_m$ proved in [MM77, Mod87], which will serve as a building block for the limit passage in the semistability condition.

Theorem 4.4 ([MM77, Mod87]) *Let $(\zeta_m)_m \subset H^1(\Gamma_{\mathrm{C}})$ fulfill*

$$\sup_{m\in\mathbb{N}} \mathcal{G}_m(\zeta_m) < \infty. \tag{4.12}$$

Then, the sequence $(\zeta_m)_m$ is precompact in $L^1(\Gamma_{\mathrm{C}})$ and every limit point belongs to $\mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0,1\})$. Moreover, the functionals $(\mathcal{G}_m)_m$ Γ-converge in $L^1(\Gamma_{\mathrm{C}})$ as $m \to \infty$ to the functional $\mathcal{G}_{\mathrm{b}}$ (3.19), viz.

Γ-lim inf ***inequality:*** *for all* $\zeta \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ *and* $(\zeta_m)_m \subset H^1(\Gamma_{\mathrm{C}})$ *with* $\zeta_m \to \zeta$ *in* $L^1(\Gamma_{\mathrm{C}})$ *there holds*

$$\liminf_{m\to\infty} \mathcal{G}_m(\zeta_m) \geq \mathcal{G}_{\mathrm{b}}(\zeta); \tag{4.13}$$

Γ-lim sup ***inequality:*** *for every* $\zeta \in \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ *there exists* $(\zeta_m)_m \subset H^1(\Gamma_{\mathrm{C}})$ *with* $\zeta_m \to \zeta$ *in* $L^1(\Gamma_{\mathrm{C}})$ *and* $\limsup_{m\to\infty} \mathcal{G}_m(\zeta_m) \leq \mathcal{G}_{\mathrm{b}}(\zeta)$.

Anyhow, let us observe that Thm. 4.4 is not sufficient to pass to the limit in the semistability condition. This is ultimately due to the fact that the rate-independent delamination process is non-static. Hence, taking the limit of (3.29) as $m \to \infty$ requires the construction of a sequence which *mutually* recovers

$$\underbrace{\mathcal{R}_1}_{\text{“dissipation"}} + \underbrace{\mathcal{G}_m}_{\text{“static energy"}}.$$

Such a construction of the mutual recovery sequence will be carried out in Section 4.2.

4.1 Step 1: Limit passage in the momentum balance

In the following we verify that the momentum balance (3.28a) holds for the SBV-adhesive limit system. For this, we aim to take the limit $m \to \infty$ in the momentum balance (3.28a) for the Modica-Mortola adhesive systems. But as (4.6a) only guarantees weak $W^{1,p}$-convergence of the Modica-Mortola adhesive displacements $(u_m)_m$, we cannot directly pass to the limit with the term of p-growth, i.e. with $\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathrm{D}W_p(e(u_m(t)){:}e(v-u_m(t))\,\mathrm{d}x$. In order to circumvent this difficulty we are going to make use of the equivalent subdifferential inclusions (3.39). For every $m \in \mathbb{N}$ and a.a. $t \in (0,T)$, these involve the elements $\ell_m(t) \in \partial_u \mathcal{J}_{\mathrm{C}}(u_m(t))$, $\mathcal{J}_{\mathrm{C}}$ from (3.35), with $(u_m(t), w_m(t), z_m(t), \ell_m(t))$ fulfilling (3.39) for a.a. $t \in (0,T)$. Here, a comparison of the terms in (3.39) together with estimates (3.47), (3.49), (3.52) yields a uniform bound for the sequence $(\ell_m)_m \subset L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*)$, i.e. $\sup_{m\in\mathbb{N}} \|\ell_m\|_{L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*)} \leq C$, and hence there exists $\ell \in L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*)$ such that up to a subsequence

$$\ell_m \rightharpoonup \ell \quad \text{in } L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)^*). \tag{4.14}$$

Moreover, due to the bound (3.47), there exists $\mu \in L^{p'}(0,T;L^{p'}(\Omega))$ such that, up to the extraction of a further (not relabeled) subsequence there holds

$$\mathrm{DW}_p(e(u_m)) \rightharpoonup \mu \quad \text{in } L^{p'}(0,T;L^{p'}(\Omega)). \tag{4.15}$$

Thus, exploiting the convergences (4.6a), (4.6j), (4.14) and (4.15) as $m \to \infty$, we obtain that the quadruple (u, w, μ, ℓ) fulfills

$$\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathrm{DR}_2(e(\dot{u}(t)))+\mathrm{DW}_2(e(u(t)))-\mathbb{B}\Theta(w(t))+\mu(t)\big){:}e(v)\,\mathrm{d}x + \int_{\Gamma_{\mathrm{C}}} kz(t)\,[\![u(t)]\!]\cdot[\![v]\!]\,\mathrm{d}S + \langle \ell(t), v\rangle = \langle \mathrm{F}(t), v\rangle \tag{4.16}$$

for all $v \in W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ and a.a. $t \in (0,T)$. Hence, in order to conclude that (4.16) is the momentum inclusion for the SBV-adhesive limit, we have to identify

$$\mu(t) = \mathrm{DW}_p(e(u(t))) \quad \text{and} \quad \ell(t) \in \partial_u \mathcal{J}_{\mathrm{C}}(u(t)) \quad \text{for a.a. } t \in (0,T). \tag{4.17}$$

This will be deduced by exploiting a well-known result from maximal monotone operator theory (see, e.g., [Att84, p. 356, Lemma 3.57], as well as Lemma 5.4 ahead), for the maximal monotone operator

$$A := \partial\mathcal{F} \ \text{ with } \mathcal{F} : L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)) \to [0,+\infty],\ \mathcal{F}(v) := \int_0^t \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathrm{W}_p(e(v)))\,\mathrm{d}x + \mathcal{J}_{\mathrm{C}}(v)\,\mathrm{d}s. \tag{4.18}$$

Note, that the identification of the limits in (4.17) will ultimately imply the strong convergence of $(u_m)_m$ in $L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$. Hence, we may state the following result:

Proposition 4.1 (Momentum balance for the SBV-adhesive model) *Let (3.7), (3.8), (3.11), and (3.13) hold true. Keep $k \in \mathbb{N}$ fixed. Consider $(u_m, z_m, w_m)_m$ such that $(u_m, z_m, w_m) \to (u, z, w)$ as $m \to \infty$ in the sense of (4.6) and such that, for all $m \in \mathbb{N}$, the triple (u_m, z_m, w_m) satisfies the Modica-Mortola adhesive momentum inclusion (3.39). Then the limit (u, z, w) fulfills the SBV-adhesive momentum inclusion for a.a. $t \in (0, T)$ and moreover we have $u_m \to u$ even strongly in $L^p(0, T; W^{1,p}(\Omega\backslash\Gamma_C; \mathbb{R}^d))$.*

Proof: In view of (4.16) it remains to identify the limits as in (4.17). For this, we apply [Att84, p. 356, Lemma 3.57] on $A = \partial\mathcal{F}$ from (4.18); in the following we use the placeholder $X = L^p(0, T; W^{1,p}(\Omega\backslash\Gamma_C; \mathbb{R}^d))$. Consider $u_m^* \in X^*$ defined by $\langle u_m^*, v\rangle_X := \int_0^t \int_{\Omega\backslash\Gamma_C} \mathrm{DW}_p(e(u_m(s))) \colon e(v(s))\,\mathrm{d}x + \langle \ell_m(s), v(s)\rangle\,\mathrm{d}s$ for all $v \in X$. It clearly fulfills $u_m^* \in A(u_m)$ and (4.14) and (4.15) yield that $u_m^* \rightharpoonup u^*$ in X^*, with u^* defined by $\langle u^*, v\rangle_X := \int_0^t \int_{\Omega\backslash\Gamma_C} \mu(s) \colon e(v(s))\,\mathrm{d}x + \langle \ell(s), v(s)\rangle\,\mathrm{d}s$. Following [Att84, p. 356, Lemma 3.57], we now check that $\limsup_{m\to\infty}\langle u_m^*, u_m\rangle_X \leq \langle u^*, u\rangle_X$. To do so, we argue as follows. We test the reformulation (3.39) of the momentum equation satisfied by (u_m, w_m, z_m) with u_m, integrate in time, and take the $\limsup_{m\to\infty}$. Thus,

$$
\begin{aligned}
&\limsup_{m\to\infty}\int_0^t\left(\int_{\Omega\backslash\Gamma_C}\mathrm{DW}_p(e(u_m))\colon e(u_m)\,\mathrm{d}x+\langle\ell_m,u_m\rangle\right)\mathrm{d}s\\
&\leq-\liminf_{m\to\infty}\underbrace{\int_0^t\int_{\Omega\backslash\Gamma_C}\mathrm{DR}_2(e(\dot u_m))\colon e(u_m)\,\mathrm{d}x\,\mathrm{d}s}_{=\mathrm{R}_2(e(\dot u_m(t)))-\mathrm{R}_2(e(\dot u_m(0)))}-\liminf_{m\to\infty}\int_0^t\int_{\Omega\backslash\Gamma_C}\mathrm{DW}_2(e(u_m))\colon e(u_m)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\liminf_{m\to\infty}\int_0^t\int_{\Gamma_C}\tfrac{k}{2}z_m\big|[\![u_m]\!]\big|^2\,\mathrm{d}S\,\mathrm{d}s+\limsup_{m\to\infty}\int_0^t\int_{\Omega\backslash\Gamma_C}\mathbb{B}\Theta(w_m)\colon e(u_m)\,\mathrm{d}x\,\mathrm{d}s+\limsup_{m\to\infty}\int_0^t\langle\mathrm{F},u_m\rangle\,\mathrm{d}s\\
&\leq-\int_0^t\int_{\Omega\backslash\Gamma_C}\mathrm{DR}_2(e(\dot u))\colon u\,\mathrm{d}x\,\mathrm{d}s-\int_0^t\int_{\Omega\backslash\Gamma_C}\big(\mathrm{DW}_2(e(u))-\mathbb{B}\Theta(w)\big)\colon e(u)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad-\int_0^t\int_{\Gamma_C}\tfrac{k}{2}z\big|[\![u]\!]\big|^2\,\mathrm{d}S\,\mathrm{d}s+\int_0^t\langle\mathrm{F},u\rangle\,\mathrm{d}s\\
&=\int_0^t\left(\int_{\Omega\backslash\Gamma_C}\mu\colon e(u)\,\mathrm{d}x+\langle\ell,u\rangle\right)\mathrm{d}s,
\end{aligned}
\tag{4.19}
$$

where the second inequality follows from convergences (4.6a), (4.6b), (4.6c), (4.6j), and the last equality from (4.16). Thus, we have $u^* \in A(u)$ by [Att84, p. 356, Lemma 3.57] and the sum rule (3.38) for $A = \partial\mathcal{F}$ yields that there exists $\ell' \in X$ with $\ell'(s) \in \partial\mathcal{J}_C(u(s))$ for a.a. $s \in (0, t)$, such that

$$
\langle u^*, v\rangle_X = \int_0^t\int_{\Omega\backslash\Gamma_C}\mu(s)\colon e(v(s))\,\mathrm{d}x+\langle\ell(s),v(s)\rangle\,\mathrm{d}s=\int_0^t\int_{\Omega\backslash\Gamma_C}\mathrm{DW}_p(u(s))\colon e(v(s))+\langle\ell'(s),v(s)\rangle\,\mathrm{d}s \tag{4.20}
$$

for all $v \in X$. We conclude that $\ell = \ell'$ and (4.17) by the fundamental lemma of the calculus of variations when choosing $v(s, x) := \varphi(s)v(x)$ for any $\varphi \in \mathrm{C}_0^\infty(0, t)$ and any $v \in W^{1,p}(\Omega\backslash\Gamma_C; \mathbb{R}^d)$. Thus, inserting this in (4.16), we find that the triple (u, w, z) complies with (3.39), and hence (3.28a) holds true. ■

4.2 Step 2: Limit passage in the semistability condition

We now prove that the pair (u, z) complies with the semistability condition (3.29) for *any* test function $\tilde z \in \mathcal{Z}_{\mathrm{SBV}} = \mathrm{SBV}(\Gamma_C; \{0, 1\})$. To do so, we follow a well-established procedure in the analysis of rate-independent systems. Viz., we prove that for all $t \in (0, T]$ there exists a *mutual recovery sequence* (or MRS, for short) $(\tilde z_m)_m \subset \mathcal{Z}_{\mathrm{MM}}$ (whose dependence on t is omitted) such that $\tilde z_m \to \tilde z$ in $L^1(\Gamma_C)$ as $m \to \infty$, and

$$
\begin{aligned}
&\limsup_{m\to\infty}\left(\Phi_{k,m}(u_m(t),\tilde z_m)+\mathcal{R}_1(\tilde z_m-z_m(t))-\Phi_{k,m}(u_m(t),z_m(t))\right)\\
&\leq\Phi_k(u(t),\tilde z)+\mathcal{R}_1(\tilde z-z(t))-\Phi_k(u(t),z(t)).
\end{aligned}
\tag{4.21}
$$

Since $\Phi_{k,m}(u_m(t),\tilde{z}_m)+\mathcal{R}_1(\tilde{z}_m{-}z_m(t)) - \Phi_{k,m}(u_m(t),z_m(t)) \geq 0$ for all $m\in\mathbb{N}$ and all $t\in[0,T]$ in view of the semistability (3.29) for the Modica-Mortola solutions (u_m,z_m), from (4.21) we will immediately deduce the desired semistability for the limit functions (u,z).

Proposition 4.2 (Mutual recovery sequences for the SBV-adhesive systems) *Let* (3.7)*,* (3.8)*,* (3.11)*, and* (3.13) *hold true. Keep $k\in\mathbb{N}$ fixed. Let $\Phi_{m,k}$ and Φ_k be given by* (3.16) *and* (3.18)*. Let $(u_m)_m$ satisfy* (4.6a) *and let $(z_m)_m\subset \mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$ with z_m semistable for $\Phi_{m,k}(u_m,\cdot)$ and $z_m \overset{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\})$. Then, for every $\tilde{z}\in\mathcal{Z}_{\mathrm{SBV}}$ there is a sequence $(\tilde{z}_m)_m\subset\mathcal{Z}_{\mathrm{MM}}$ with $\tilde{z}_m\to\tilde{z}$ in $L^1(\Gamma_{\mathrm{C}})$ such that* (4.21) *holds.*

Proof: We draw the definition of the MRS $(\tilde{z}_m)_m$ from the proof of [Tho11, Lemma 3.5] and, for the reader's convenience, we outline here the main steps in the construction, referring to [Tho11] for all details. We suppose that $\Phi_k(u(t),\tilde{z})<\infty$ and $\mathcal{R}_1(\tilde{z}{-}z(t))<\infty$ (otherwise, the recovery sequence is trivial). For (4.21) to hold, it is also necessary that $\Phi_{k,m}(u_m(t),\tilde{z}_m)<\infty$ and $\mathcal{R}_1(\tilde{z}_m{-}z_m(t))<\infty$. Therefore, in [Tho11] the construction from the proof of Thm. 4.4 in [MM77, Mod87] is suitably adapted to accomodate the latter constraint. Viz., one sets

$$\tilde{z}_m := \max\{0,\min\{(\hat{z}_m{-}\delta_m), z_m(t)\}\} \quad \text{with } \delta_m := \|\hat{z}_m{-}\tilde{z}{+}z(t){-}z_m(t)\|^{1/2}_{L^1(\Gamma_{\mathrm{C}})}. \tag{4.22}$$

Here, $(\hat{z}_m)_m$ is the classical recovery sequence used in [MM77, Mod87] to prove the Γ-$\limsup$ condition of Thm. 4.4. In particular, this sequence

$$(\hat{z}_m)_m\subset L^1(\Gamma_{\mathrm{C}}) \text{ fulfills } \quad \begin{cases} \hat{z}_m\to\tilde{z} \ \text{ in } L^1(\Gamma_{\mathrm{C}}),\\ \limsup_{m\to\infty}\mathcal{G}_m(\hat{z}_m)\leq\mathcal{G}_{\mathrm{b}}(\tilde{z}). \end{cases} \tag{4.23}$$

By definition, we have $0\leq\tilde{z}_m\leq z_m(t)\leq 1$ a.e. on Γ_{C}. It follows from (4.23) and (4.6e) that $\delta_m\to0$. Exploiting this, it can be shown that $\tilde{z}_m\to\tilde{z}$ in $L^1(\Gamma_{\mathrm{C}})$, hence $\mathcal{R}_1(\tilde{z}_m{-}z_m(t))\to\mathcal{R}_1(\tilde{z}{-}z(t))$. Since $(\tilde{z}_m)_m$ is bounded in $L^\infty(\Gamma_{\mathrm{C}})$, we immediately have

$$\tilde{z}_m\to\tilde{z} \quad \text{in } L^q(\Gamma_{\mathrm{C}}) \text{ for all } 1\leq q<\infty. \tag{4.24}$$

Combining (4.6e), (4.7), and (4.24), we then infer

$$\begin{cases} \lim_{m\to\infty}\int_{\Gamma_{\mathrm{C}}}\frac{k}{2}\,(\tilde{z}_m{-}z_m(t))\,|[\![u_m(t)]\!]|^2\,\mathrm{d}S = \int_{\Gamma_{\mathrm{C}}}\frac{k}{2}\,(\tilde{z}{-}z(t))\,|[\![u(t)]\!]|^2\,\mathrm{d}S,\\ \lim_{m\to\infty}\int_{\Gamma_{\mathrm{C}}}a_0(z_m(t){-}\tilde{z}_m)\,\mathrm{d}S = \int_{\Gamma_{\mathrm{C}}}a_0(z(t){-}\tilde{z})\,\mathrm{d}S. \end{cases} \tag{4.25}$$

Repeating the very same calculations as in the proof of [Tho11, Lemma 3.5], one can also show that

$$\limsup_{m\to\infty}\,(\mathcal{G}_m(\tilde{z}_m){-}\mathcal{G}_m(z_m(t)))\leq\mathcal{G}_{\mathrm{b}}(\tilde{z})-\mathcal{G}_{\mathrm{b}}(z(t)).$$

This concludes the proof of (4.21). ■

4.3 Bonus: energy and enthalpy equalities in the adhesive case

In the following we establish that the mechanical energy (3.30), the enthalpy (3.32) and the total energy (3.34) inequalities hold even as equalities in the adhesive setting, cf. Theorem 3.1. For the proof, we will confine ourselves to the SBV-adhesive system, but let us stress that the respective *equalities* indeed hold for the Modica-Mortola adhesive system and they can be proved along the same lines as in what follows. In the Modica-Mortola framework, the approximating systems to be used in the subsequent steps are the time-discrete systems outlined in the Appendix A.1, to which we refer for more details.

We start with proving the opposite relation for the mechanical energy inequality (3.30) of the SBV-adhesive system. In [Rou10, RR11b] this was obtained by applying a Riemann-sum argument on the semistability inequality and by testing the momentum balance by the solution $\dot{u}$ of the adhesive system. In our setting, however, the momentum balance cannot be tested by $\dot{u}$, as test functions are required to have $W^{1,p}$-regularity in $\Omega\backslash\Gamma_{\rm C}$, cf. (3.23). To avoid testing with $\dot u$, we adopt the Riemann-sum technique also for the momentum balance: For an equidistant partition of the interval $[0,T]$,

$$0 = t_0 < t_1^N < \ldots < t_N^N = T \qquad \text{with} \qquad t_i^N - t_{i-1}^N = \tau_N\,, \tag{4.26}$$

we test the adhesive momentum inclusion (3.39) at time t_{i-1}^N by the differences $u_i^N - u_{i-1}^N$

$$\begin{aligned}\int_{\Omega\backslash\Gamma_{\rm C}} \big(\mathrm{D}\mathcal{R}_2(e(\dot u_{i-1}^N))+\mathrm{D}W_2(e(u_{i-1}^N)) - \mathbb{B}\Theta_{i-1}^N + \mathrm{D}W_p(e(u_{i-1}^N))\big):e(u_i^N - u_{i-1}^N)\,\mathrm{d}x\\ + \langle\lambda_{i-1}^N, u_i^N - u_{i-1}^N\rangle\langle F_i^N, u_i^N - u_{i-1}^N\rangle\end{aligned} \tag{4.27}$$

with $\lambda_{i-1}^N \in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*$ s.t. $\lambda_{i-1}^N \in \partial\mathcal{F}_k(u_{i-1}^N, z_{i-1}^N)$; here we abbreviated $u_i^N := u(t_i^N)$, etc.. Then we will exploit convexity inequalities for W_2, W_p and λ_{i-1}^N. Let us point out that the semistability condition is valid for all $t \in [0,T]$, whereas the momentum balance (3.28a) holds only for *almost every* $t \in (0,T)$. Hence, the sequence of partitions $(\tau_N)_N$ with $\tau_N \to 0$ as $N \to \infty$ has to be carefully chosen such that (3.28a), i.e. (4.27) holds for every t_i^N involved.

Proposition 4.3 (Upper estimate for the mechanical energy) *Let* (3.7), (3.8), (3.11), *and* (3.13) *hold true. Let the energy of the adhesive contact system be given by either* $\Phi_{k,m}$ *from* (3.16) *for any* $m,k>0$ *fixed, or by* Φ_k *from* (3.18) *for any* $k>0$ *fixed. Then the mechanical energy inequality* (3.30) *also holds in the opposite direction, i.e.*

$$\begin{aligned}&\Phi\big(u(t),z(t)\big) + \int_0^t 2\,\mathcal{R}_2(e(\dot u))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_1}(z;[0,t])\\ &\geq \Phi\big(u_0,z_0\big) + \int_0^t\int_{\Omega\backslash\Gamma_{\rm C}} \Theta(w)\mathbb{B}{:}e(\dot u)\,\mathrm{d}x\mathrm{d}s + \int_0^t \langle \mathrm{F},\dot u\rangle\,\mathrm{d}s \qquad \textit{for all } t\in[0,T],\end{aligned} \tag{4.28}$$

with $\Phi \in \{\Phi_{k,m},\Phi_k\}$. *Hence, we have mechanical energy equality for the adhesive systems.*

Proof: Consider a sequence of partitions (4.26) with $\tau_N \to 0$ as $N\to\infty$, such that (4.27) is well defined for all $N\in\mathbb{N}$. Since W_2 and W_p are convex, we have

$$\begin{aligned}&\sum_{i=1}^N \int_{\Omega\backslash\Gamma_{\rm C}} \big(\mathrm{D}W_2(e(u_{i-1}^N)) + \mathrm{D}W_p(e(u_{i-1}^N))\big):e(u_i^N - u_{i-1}^N)\,\mathrm{d}x\\ &\leq \sum_{i=1}^N \int_{\Omega\backslash\Gamma_{\rm C}} \big(W_2(e(u_i^N))+W_p(e(u_i^N)) - W_2(e(u_{i-1}^N))-W_p(e(u_{i-1}^N))\big)\,\mathrm{d}x\\ &= \int_{\Omega\backslash\Gamma_{\rm C}} \big(W_2(e(u(T)))+W_p(e(u(T))) - W_2(e(u_0))-W_p(e(u_0))\big)\,\mathrm{d}x\end{aligned} \tag{4.29}$$

For the right-hand side of (4.27) we obtain

$$\sum_{i=1}^N \langle F_{i-1}^N, u_i^N - u_{i-1}^N\rangle = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \langle F(s),\dot u(s)\rangle + \underbrace{\langle F_{i-1}^N - F(s),\dot u(s)\rangle}_{\to 0}\,, \tag{4.30}$$

where the second term tends to 0 due to the regularity (3.11a) of F and (4.6a) of u. For all $k\in\mathbb{N}$ we have for $\lambda_{i-1}^N \in \partial\mathcal{F}_k(u_{i-1}^N)$ with $\langle\lambda_{i-1}^N, v\rangle = \langle \ell_{i-1}^N, v\rangle + \int_{\Gamma_{\rm C}} kz_{i-1}^N[\![u_{i-1}^N]\!]{\cdot}[\![v]\!]\,\mathrm{d}S$ and $\ell_{i-1}^N \in \partial\mathcal{J}_{\rm C}(u_{i-1}^N)$. Exploiting the convexity of $\mathcal{J}_{\rm C}$ and that $\mathcal{J}_{\rm C}(u_{i-1}^N) = \mathcal{J}_{\rm C}(u_i^N) = 0$ we find

$$\begin{aligned}&\sum_{i=1}^N \langle \ell_{i-1}^N, u_i^N - u_{i-1}^N\rangle + \sum_{i=1}^N \int_{\Gamma_{\rm C}} kz_{i-1}^N[\![u_{i-1}^N]\!]\cdot[\![u_i^N - u_{i-1}^N]\!]\,\mathrm{d}S \leq 0 + \sum_{i=1}^N \int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\rm C}} kz_{i-1}^N[\![u_{i-1}^N]\!]\cdot[\![\tfrac{u_i^N - u_{i-1}^N}{\tau_N}]\!]\,\mathrm{d}S\,\mathrm{d}s\\ &= \underbrace{\sum_{i=1}^N \int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\rm C}} kz_{i-1}^N[\![u_{i-1}^N]\!]\cdot[\![\dot u_{i-1}^N]\!]\,\mathrm{d}S\,\mathrm{d}s}_{\downarrow\ \int_0^T\int_{\Gamma_{\rm C}} kz[\![u]\!]\cdot[\![\dot u]\!]\,\mathrm{d}S\,\mathrm{d}s} + \underbrace{\sum_{i=1}^N \int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\rm C}} kz_{i-1}^N[\![u_{i-1}^N]\!]\cdot[\![\tfrac{u_i^N - u_{i-1}^N}{\tau_N} - \dot u_{i-1}^N]\!]\,\mathrm{d}S\,\mathrm{d}s}_{\downarrow\ +0}\,,\end{aligned} \tag{4.31}$$

where the convergence of the Riemann-sums is due to (4.6a) and (4.6c). To obtain that the second term on the right-hand side tends to 0 one uses that $\|z\|_{L^\infty} \leq 1$ and then applies Hölder's inequality in $L^2(0,T;L^2(\Gamma_{\rm C};\mathbb{R}^d)$ together with

$$
\begin{aligned}
\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\big\|\tfrac{u_i^N-u_{i-1}^N}{\tau_N}-\dot u_{i-1}^N\big\|^2_{W^{1,2}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}\,\mathrm{d}t &=\sum_{i=1}^{N}\tau_N\big\|\tfrac{u_i^N-u_{i-1}^N}{\tau_N}\big\|^2_{W^{1,2}}+\tau_N\|\dot u_{i-1}^N\|^2_{W^{1,2}}-2\tau_N\langle\tfrac{u_i^N-u_{i-1}^N}{\tau_N},\dot u_{i-1}^N\rangle \\
&\to \|\dot u\|^2_{L^2(0,T;W^{1,2})}+\|\dot u\|^2_{L^2(0,T;W^{1,2})}-2\|\dot u\|^2_{L^2(0,T;W^{1,2})}=0\,,
\end{aligned}
\tag{4.32}
$$

where the convergence of the Riemann-sums is due to (4.6a).

For the term involving the viscous dissipation we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega\backslash\Gamma_{\rm C}}\mathrm{DR}_2(e(\dot u_{i-1})):e(u_i^N-u_{i-1}^N)\,\mathrm{d}x\\
&=\underbrace{\sum_{i=1}^{N}\int_{\Omega\backslash\Gamma_{\rm C}}(t_i^N-t_{i-1}^N)\mathrm{DR}_2(e(\dot u_{i-1})):e(\dot u_{i-1}))\,\mathrm{d}x}_{\downarrow\ \int_0^T\int_{\Omega\backslash\Gamma_{\rm C}}\mathrm{DR}_2(e(\dot u)):e(\dot u)\,\mathrm{d}x\,\mathrm{d}s}+\underbrace{\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Omega\backslash\Gamma_{\rm C}}\mathrm{DR}_2(e(\dot u_{i-1})):e(\tfrac{u_i^N-u_{i-1}^N}{\tau_N}-\dot u_{i-1})\,\mathrm{d}x\,\mathrm{d}s}_{\downarrow\ +0\,,}
\end{aligned}
\tag{4.33}
$$

where the convergence of the Riemann-sums is again due to (4.6a) and the convergence to 0 of the second term is obtained using (4.32). It remains to analyze the term involving the thermal stresses.

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega\backslash\Gamma_{\rm C}}-\mathbb{B}\Theta_{i-1}:e(u_i^N-u_{i-1}^N)\,\mathrm{d}x\\
&=\underbrace{\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Omega\backslash\Gamma_{\rm C}}-\mathbb{B}\Theta_{i-1}:e(\dot u_{i-1}^N)\,\mathrm{d}x\,\mathrm{d}s}_{\downarrow\ \int_0^T\int_{\Omega\backslash\Gamma_{\rm C}}-\mathbb{B}\Theta(w):e(\dot u)\,\mathrm{d}x\,\mathrm{d}s}+\underbrace{\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Omega\backslash\Gamma_{\rm C}}-\mathbb{B}\Theta_{i-1}:e(\tfrac{u_i^N-u_{i-1}^N}{\tau_N}-\dot u_{i-1}^N)\,\mathrm{d}x\,\mathrm{d}s}_{\downarrow\ +0\,,}
\end{aligned}
\tag{4.34}
$$

where we exploited (4.6a), (4.6j), and again (4.32). Collecting (4.29)–(4.34) leads to

$$
\begin{aligned}
\int_0^T\langle F,\dot u\rangle\,\mathrm{d}s\leq&\int_{\Omega\backslash\Gamma_{\rm C}}\big(W_2(e(u(T)))+W_p(e(u(T)))-W_2(e(u_0))-W_p(e(u_0))\big)\,\mathrm{d}x\\
&+\int_0^T\int_{\Omega\backslash\Gamma_{\rm C}}2\mathrm{R}_2(e(\dot u))-\mathbb{B}\Theta(w):e(\dot u)\,\mathrm{d}x\,\mathrm{d}s+\int_0^T\int_{\Gamma_{\rm C}}kz[\![u]\!]\cdot[\![\dot u]\!]\,\mathrm{d}S\,\mathrm{d}s
\end{aligned}
\tag{4.35}
$$

Now, a similar estimate for the surface energy has to be established. As in [Rou10] we therefore test the semistability inequality at time t_{i-1}^N with z_i^N. Summing up over $i\in\{0,\dots,N\}$ yields

$$
\begin{aligned}
\sum_{i=1}^{N}\int_{\Gamma_{\rm C}}\tfrac{k}{2}z_{i-1}^N\big|[\![u_{i-1}^N]\!]\big|^2\,\mathrm{d}S+\mathcal{G}_{\rm b}(z_{i-1}^N)&\leq\sum_{i=1}^{N}\int_{\Gamma_{\rm C}}\tfrac{k}{2}z_i^N\big|[\![u_{i-1}^N]\!]\big|^2\,\mathrm{d}S+\mathcal{G}_{\rm b}(z_i^N)+\mathcal{R}_1(z_i^N-z_{i-1}^N)\\
=\sum_{i=1}^{N}\int_{\Gamma_{\rm C}}\tfrac{k}{2}z_i^N\big|[\![u_i^N]\!]\big|^2\,\mathrm{d}S+\mathcal{G}_{\rm b}(z_i^N)&+\mathcal{R}_1(z_i^N-z_{i-1}^N)+\sum_{i=1}^{N}\int_{\Gamma_{\rm C}}\tfrac{k}{2}z_i^N\big(\big|[\![u_{i-1}^N]\!]\big|^2-\big|[\![u_i^N]\!]\big|^2\big)\,\mathrm{d}S\,.
\end{aligned}
\tag{4.36}
$$

Scooping the left-hand side to the right, exploiting the cancellation of redundant terms and using that the last term in (4.36) can be expressed via the chain rule, leads to

$$
\begin{aligned}
0\leq&\int_{\Gamma_{\rm C}}\tfrac{k}{2}z(T)\big|[\![u(T)]\!]\big|^2\,\mathrm{d}S-\int_{\Gamma_{\rm C}}\tfrac{k}{2}z_0\big|[\![u_0]\!]\big|^2\,\mathrm{d}S+\mathcal{G}_{\rm b}(z(T))-\mathcal{G}_{\rm b}(z_0)+\mathcal{R}_1(z(T)-z_0)\\
&-\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\rm C}}kz_i^N[\![u(s)]\!]\,[\![\dot u(s)]\!]\,\mathrm{d}S\,\mathrm{d}s\,.
\end{aligned}
\tag{4.37}
$$

For the last term in (4.37) we calculate

$$
\begin{aligned}
&-\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\mathrm{C}}} kz_i^N [\![u(s)]\!]\,[\![\dot u(s)]\!]\,\mathrm{d}S\,\mathrm{d}s\\
&\leq \underbrace{-\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\mathrm{C}}} kz_i^N [\![u_i^N]\!]\,[\![\dot u_i^N]\!]\,\mathrm{d}S\,\mathrm{d}s}_{\downarrow} + \underbrace{\sum_{i=1}^{N}\int_{t_{i-1}^N}^{t_i^N}\int_{\Gamma_{\mathrm{C}}} kz_i^N \big(\big|[\![u_i^N]\!]-[\![u]\!]\big|\big|[\![\dot u_i^N]\!]\big| + \big|[\![\dot u]\!]-[\![\dot u_i^N]\!]\big|\big|[\![u]\!]\big|\big)\mathrm{d}S\,\mathrm{d}s}_{\downarrow}\\
&\qquad -\int_0^T\int_{\Gamma_{\mathrm{C}}} kz[\![u]\!]\,[\![\dot u]\!]\,\mathrm{d}S\,\mathrm{d}s \qquad\qquad\qquad\qquad\qquad\qquad + 0\,,
\end{aligned}
$$

where above the convergence of the Riemann-sums is due to $u \in W^{1,2}(0,T;W^{1,2}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d))$ by (4.6a) and $z \in L^\infty((0,T)\times\Gamma_{\mathrm{C}}$ by (4.6c). Altogether we have obtained

$$
0 \leq \int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2} z(T)\big|[\![u(T)]\!]\big|^2\,\mathrm{d}S - \int_{\Gamma_{\mathrm{C}}} \tfrac{k}{2} z_0\big|[\![u_0]\!]\big|^2\,\mathrm{d}S + \mathcal{G}_{\mathrm{b}}(z(T)) - \mathcal{G}_{\mathrm{b}}(z_0) + \mathcal{R}_1(z(T)-z_0) - \int_0^T\int_{\Gamma_{\mathrm{C}}} kz[\![u]\!]\,[\![\dot u]\!]\,\mathrm{d}S\,\mathrm{d}s\,. \tag{4.38}
$$

The bulk (4.35) and the surface (4.38) estimates yield the upper mechanical energy estimate (4.28) for the SBV-adhesive system. The same result is obtained for the Modica-Mortola adhesive system, when repeating the steps for the surface energy with the regularization $\mathcal{G}_m$ instead of $\mathcal{G}_{\mathrm{b}}$ in (4.36)–(4.38). ■

With the aid of the mechanical energy equality one is able to conclude the convergence of the viscous dissipation

$$
\int_0^t 2\mathcal{R}_2(e(\dot u_m))\,\mathrm{d}s \to \int_0^t 2\mathcal{R}_2(e(\dot u))\,\mathrm{d}s, \tag{4.39}
$$

where (u_m, z_m, w_m) are the solutions of the Modica-Mortola adhesive systems and (u, z, w) is the solution of the SBV-adhesive system. Therefore, we can state the following

Corollary 4.4 (Enthalpy and total energy equality) *Let the assumptions of Theorem 4.3 hold. Then the enthalpy* (3.32) *and the total energy* (3.34) *estimates hold as equalities for the adhesive systems.*

Proof: To prove (4.39), as in [Rou10, RR11b] we develop the following chain of inequalities

$$
\begin{aligned}
&\int_0^t 2\mathcal{R}_2(e(\dot u))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_1}(z;[0,t]) \leq \liminf_{m\to\infty}\int_0^t 2\mathcal{R}_2(e(\dot u_m))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_1}(z_m;[0,t])\\
&\leq \limsup_{m\to\infty} \Phi_{m,k}(u_m^0, z_m^0) - \Phi_{m,k}(u_m(t), z_m(t)) + \int_0^t\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \Theta(w_m)\mathbb{B}:e(\dot u_m)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle F, \dot u_m\rangle\,\mathrm{d}s\\
&= \Phi_k(u(0), z(0)) - \Phi_k(u(t), z(t)) + \int_0^t\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \Theta(w)\mathbb{B}:e(\dot u)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle F, \dot u\rangle\,\mathrm{d}s\\
&= \int_0^t 2\mathcal{R}_2(e(\dot u))\,\mathrm{d}s + \mathrm{Var}_{\mathcal{R}_1}(z;[0,t])\,.
\end{aligned} \tag{4.40}
$$

Here, the first inequality is obtained by lower semicontinuity and the convergences (4.6a), (4.6c), while the second one relies on the mechanical energy equality for the Modica-Mortola adhesive systems. The third equality is due to $u_m \to u$ strongly in $L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$ by Thm. 4.1, assumption (4.5) and convergence (4.6j), whereas the mechanical energy equality for the SBV-adhesive systems is exploited for the last equality. Since $\mathrm{Var}_{\mathcal{R}_1}(z_m;[0,t]) \to \mathrm{Var}_{\mathcal{R}_1}(z;[0,t])$ by (4.6c), from (4.40) we deduce the convergence of the viscous dissipation. Moreover, with the same arguments as in [Rou10] one can conclude that $\mathrm{R}_1(\dot z_m) \overset{*}{\rightharpoonup} \xi_{\dot z}^{\mathrm{surf}}$ in measure on $\overline{(0,T)\times\Gamma_{\mathrm{C}}}$. Arguing along the lines of [RR11b, p. 3186] these observations together with convergences (4.6a), (4.6c) and (4.6j) allow us to pass with $m \to \infty$ in the weak enthalpy *equality* of the Modica-Mortola adhesive systems and to obtain that the limit, i.e. the respective relation for the SBV-adhesive system, again is an *equality.* Finally, the total energy equality for the SBV-adhesive system is deduced by summing up the mechanical energy and the enthalpy equality. ■

For the brittle delamination systems, however, our methods to gain energy equalities fail in the very first step, as the following remark highlights.

Remark 4.5 (Failure of the methods in the brittle setting) As described along with (4.27) we have to avoid the occurrence of $\dot{u}$ in nonlinear, p-dependent terms due to a lack of regularity. In the adhesive setting we therefore test the momentum inclusion at time t_{i-1}^N by $u(t_i^N)-u(t_{i-1}^N)$ and exploit convexity inequalities for W_p and $\mathcal{I}_\mathrm{C}$. In this step, to get the missing estimate, it is crucial to test with $u(t_i^N)-u(t_{i-1}^N)$. In the brittle setting however, $u(t_i^N)$ is not a suitable test function at time t_{i-1}^N since $z(t_i^N)[\![u(t_i^N)]\!]\neq 0$ is not excluded a.e. on Γ_C. Clearly this problem does not occur in the adhesive setting.

5 From SBV-adhesive contact to SBV brittle delamination

In this section we deduce the existence of energetic solutions for the SBV-brittle delamination systems. This will be done by passing to the brittle limit $k\to\infty$ with the SBV-adhesive contact systems.

During the limit passage as $k\to\infty$ the properties of the surface energy functionals $\mathcal{F}_k$ from (3.37) change dramatically: their *smooth* contributions $\mathcal{J}_k(\cdot,z_k)$ for adhesive contact from (3.36) are supposed to approximate the *nonsmooth* functionals $\mathcal{J}_\infty(\cdot,z)$ for the brittle constraint from (3.40). In addition, also a suitable convergence of their functional derivatives is required in order to pass to the limit in the weak formulation of the momentum balance, see (3.28a) and (3.28b), respectively.

Testing the adhesive momentum balance (3.28a) with functions suited for the brittle equation (3.28b), i.e. functions in the set $\mathcal{U}_{z(t)}$ from (3.24), would need

$$\text{for all } v\in\mathcal{U}_{z(t)}:\quad \int_{\Gamma_\mathrm{C}} kz_k(t)[\![u_k(t)]\!]\cdot[\![v]\!]\,\mathrm{d}x \quad \stackrel{!}{\longrightarrow} 0 \quad \text{as } k\to\infty \tag{5.1}$$

for a.a. $t\in(0,T)$, where $(u_k,z_k,w_k)_k$ are the SBV-adhesive solutions suitably converging to a limit (u,z,w). But as we only have that $\int_{\Gamma_\mathrm{C}} kz_k(t)|[\![u_k(t)]\!]|^2\mathrm{d}S\leq C$, while $\int_{\Gamma_\mathrm{C}} z_k(t)|[\![v]\!]|^2\mathrm{d}S\to 0$ only without the prefactor k, the integral in (5.1) might even blow up to ∞. Hence, we have to avoid dealing with (5.1), i.e. passing to the limit in (3.28a) with fixed test functions $v\in\mathcal{U}_{z(t)}$. Instead, we intend to construct a suitable recovery sequence $(v_k)_k$ for the test functions $v\in\mathcal{U}_{z(t)}$, which satisfies

$$\mathcal{J}_k(v_k,z_k(t))=\int_{\Gamma_\mathrm{C}}\tfrac{k}{2}z_k(t)\big|[\![v_k]\!]\big|^2\,\mathrm{d}x=0 \quad \text{for all } k\in\mathbb{N} \text{ and for all } t\in[0,T]\,. \tag{5.2}$$

Additionally $(v_k)_k$ has to feature a convergence suited to recover the bulk terms. In other words, for every $k\in\mathbb{N}$, v has to be modified in such a way that the support of $[\![v_k]\!]$ fits to the null set of z_k and, as $k\to\infty$, also $v_k\to v$ suitably in the bulk. For obvious reasons, this convergence necessitates that the supports of z_k converge to the support of z in the sense that for a.a. $t\in(0,T)$ it holds

$$\operatorname{supp} z_k(t)\subset\operatorname{supp} z(t)+B_{\rho(k,t)}(0) \text{ for all } k\in\mathbb{N} \quad\text{and}\quad \rho(k,t)\to 0 \text{ as } k\to\infty\,, \tag{5.3}$$

where $B_{\rho(k,t)}(0)$ is the open ball around 0 of radius $\rho(k,t)$; the above inclusion has to be understood as $\mathcal{L}^{d-1}(\operatorname{supp} z_k(t)\backslash(\operatorname{supp} z(t)+B_{\rho(k,t)}(0)))=0$. This so-called *support convergence* cannot be deduced from the convergence of functions in a particular metric. It is rather a fine property of sequences being *semistable* for the perimeter functional, as we will establish in Section 6.

Nonetheless, apart from this, the convergence of the bulk terms requires $v_k\to v$ *strongly* in the respective Sobolev space over the domain $\Omega_-\cup\hat{M}\cup\Omega_+$ with $\hat{M}=\operatorname{supp} z(t)$. The strong convergence of the recovery sequence can be gained from a result in [Lew88], which, for general $\hat{M}$ (of bad regularity), is only valid in $W^{1,p}(\Omega_-\cup\hat{M}\cup\Omega_+;\mathbb{R}^d)$ with $p>d$. This is the ultimate reason for the regularization of p-growth in the bulk energy. Like in Section 4.1, we cannot directly pass to the limit with the term of p-growth in the momentum inequality (3.28a), i.e. here with $\int_{\Omega\backslash\Gamma_\mathrm{C}}\mathrm{D}W_p(e(u_k(t))\!:\!e(v_k-u_k(t))\,\mathrm{d}x$, as we again have to identify the weak limit of the sequence $(\mathrm{D}W_p(e(u_k)))_k$. In fact, we will rather use the construction of $(v_k)_k$ to show that the sequence of functionals $(\mathcal{F}_k)_k$ from (3.37) Mosco-converges to the functional $\mathcal{F}_\infty$ from (3.41), cf. Prop. 5.3 in Sec. 5.1. This will allow us to conclude G-convergence of the corresponding maximal monotone subdifferential operators and hence, to carry out the limit passage in the equivalent subdifferential reformulation (3.39). For the reader's convenience, we recall the following definition, see e.g. [Att84, Sec. 3.3, p. 295].

Definition 5.1 (Mosco-convergence) *Let X be a Banach space and consider the functionals $F_k : X \to \mathbb{R}_\infty$, and $F : X \to \mathbb{R}_\infty$. We say that the sequence $(F_k)_k$* Mosco*-converges as $k \to \infty$ to the functional F, if the following two conditions hold:*
– lim inf ***inequality:*** *for every $u \in X$ and $(u_k)_k \subset X$ there holds*

$$u_k \rightharpoonup u \quad \text{weakly in } X \ \Rightarrow\ \liminf_{k\to\infty} F_k(u_k) \geq F(u); \tag{5.4}$$

– lim sup ***inequality:*** *for every $v \in X$ there exists a sequence $(v_k)_k \subset X$ such that*

$$v_k \to v \quad \text{strongly in } X \text{ and } \limsup_{k\to\infty} F_k(v_k) \leq F(v). \tag{5.5}$$

A closely related concept is the one of G-convergence of a sequence of maximal monotone operators: by [Att84, p. 373, Thm. 3.66], Mosco-convergence of the functionals implies the one of the corresponding subdifferential operators. We recall (see [Att84, p. 360, Def. 3.58]) that, given A_k, $A : X \rightrightarrows X^*$ (set-valued) maximal monotone operators defined on a Banach space X,

$$(A_k)_k\ G\text{-converges in } X \text{ to } A \ \Leftrightarrow\ \begin{cases} \forall\,(u,u^*)\in X\times X^* \text{ with } u^*\in A(u), \\ \exists\,(u_k,u_k^*)_k \text{ with } (u_k,u_k^*)\in X\times X^* \text{ and } u_k^*\in A_k(u_k): \\ u_k \to u \text{ strongly in } X, \quad u_k^* \to u^* \text{ strongly in } X^*. \end{cases} \tag{5.6}$$

After outlining the features of our approach, let us now state the **main result** of this paper.

Theorem 5.1 (Adhesive contact approximation of SBV-brittle delamination) *Assume* (3.7), (3.8) *and* (3.11). *Let $(u_k, w_k, z_k)_k$ be a sequence of* approximable *solutions of the* SBV*-adhesive contact system, supplemented with initial data $(u_k^0, \theta_k^0, z_k^0)_k$ fulfilling* (3.13) *and* (4.1). *Suppose that, as $k \to \infty$*

$$u_k^0 \rightharpoonup u_0 \quad \text{in } W^{1,p}(\Omega\backslash\Gamma_C;\mathbb{R}^d), \quad \theta_k^0 \to \theta_0 \quad \text{in } L^{\omega_1}(\Omega), \quad z_k^0 \overset{*}{\rightharpoonup} z_0 \quad \text{in } L^\infty(\Gamma_C), \text{ and} \tag{5.7}$$

$$\Phi_k(u_k^0, z_k^0) \to \Phi_b(u_0, z_0). \tag{5.8}$$

Then, there exist a (not relabeled) subsequence, and a triple (u, w, z), such that convergences (4.6) *hold for (u_k, w_k, z_k) as $k \to \infty$ and (u, w, z) is an* energetic solution *to the* SBV*-brittle delamination system, fulfilling the semistability condition* (3.29) *for all $t \in [0, T]$. In addition we have that*

$$u_k \to u \text{ in } L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_C;\mathbb{R}^d)) \quad \text{and} \quad \Phi_k(u_k, z_k) \to \Phi_b(u, z)\,. \tag{5.9}$$

Furthermore, (4.2) *holds.*

Proof: The proof follows the scheme outlined in Section 3.4.

Step 0 : **selection of converging subsequences.** For the sequence $(u_k, w_k, z_k)_k$, estimates (3.47)–(3.52) are valid and thus convergences (4.6) can be obtained in the very same way as in the proof of Thm. 4.3. Furthermore, notice that

$$\sup_{k\in\mathbb{N}}\sup_{t\in[0,T]} \Phi_k(u_k(t), z_k(t)) \leq C \ \Rightarrow\ \frac{k}{2}\int_{\Gamma_C} z_k(t)|[\![u_k(t)]\!]|^2\,\mathrm{d}S \leq C \text{ for all } t \in [0,T],\ k \in \mathbb{N}. \tag{5.10}$$

Now, it follows from (4.6b) via Sobolev trace theorems that $[\![u_k]\!] \to [\![u]\!]$ in $\mathrm{C}^0([0,T];\mathrm{C}^0(\Gamma_C;\mathbb{R}^d))$. Hence, we obtain $[\![u]\!]{\cdot}\mathrm{n} \geq 0$, and also taking into account (4.6d) we find that $\int_{\Gamma_C} z_k(t)|[\![u_k(t)]\!]|^2\,\mathrm{d}S \to \int_{\Gamma_C} z(t)|[\![u(t)]\!]|^2\,\mathrm{d}S$ for all $t \in [0,T]$. Therefore, on account of (5.10) we easily conclude that the limit pair (u, z) fulfills the brittle constraint $z[\![u]\!] = 0$ a.e. on $(0,T) \times \Gamma_C$.

The proof of **Steps** 1 **and** 2, **momentum balance and semistability**, will be carried out in Sections 5.1 and 5.2, respectively. The mechanical energy inequality (3.30) and the enthalpy inequality (3.32) can be obtained by the very same lower semicontinuity arguments as in **Step** 3 and **Step** 4 of the proof of Thm. 4.3, and the same for the positivity of the temperature, that is why we do not repeat it. ∎

5.1 Step 1: limit passage in the momentum equation via recovery sequences

In this section we pass from adhesive to brittle in the subdifferential formulations of the momentum balance. As already mentioned, this will be done with the aid of a recovery sequence $(v_k)_k$ for the test functions $v \in \mathfrak{U}_{z(t)}$ of the brittle momentum balance, which has to satisfy (5.2). The construction of this recovery sequence relies on the following Proposition 5.2. It was developed in [MRT12, Cor. 4.10] in order to pass from (Sobolev-) gradient delamination to Griffith-type delamination in the rate-independent setting. Its proof is based on a Hardy inequality derived in [Lew88, p. 190], which requires the assumption $p > d$.

In this section we will often indicate that $x = (x_1, y) \in \Omega$ is composed of the x_1-component and $y := (x_2, \ldots, x_d) \subset \mathbb{R}^{d-1}$. Moreover, in view of assumption (3.7c), we suppose without loss of generality that Ω is rotated in such a way that the normal n on Γ_{C} points in the x_1-direction.

Proposition 5.2 (Recovery sequence for the test functions, [MRT12])
Keep $t \in [0,T]$ fixed. Let $z(t) \in L^\infty(\Gamma_{\mathrm{C}})$ and let $\hat{M}(t) := \operatorname{supp} z(t)$. Let $d_{\hat{M}}(t,x) := \min_{\hat{x}\in\hat{M}(t)} |x - \hat{x}|$ for all $x \in \overline{\Omega_\pm}$. Let $v(t) \in W^{1,p}(\Omega_- \cup \hat{M}(t) \cup \Omega_+, \mathbb{R}^d)$, with $p > d$, such that $v(t) = 0$ on Γ_{Dir} in the trace sense. With $\xi_\rho^{\hat{M}}(t,x) := \min\{\frac{1}{\rho(t)}(d_{\hat{M}}(t,x) - \rho(t))^+, 1\}$ set

$$v^\rho(t, x_1, y) := v_{\mathrm{sym}}(t, x_1, y) + \xi_\rho^{\hat{M}}(t, x_1, y)\, v_{\mathrm{anti}}(t, x_1, y)\,, \tag{5.11}$$

where $v_{\mathrm{sym}}(t,x_1,y) := \frac{1}{2}(v(t,x_1,y) + v(t,-x_1,y))$ and $v_{\mathrm{anti}}(t,x_1,y) := \frac{1}{2}(v(t,x_1,y) - v(t,-x_1,y))$. Then, for a.a. $t \in [0,T]$ the following statements hold:

(i) $v^\rho(t) \to v(t)$ strongly in $W^{1,p}(\Omega_- \cup \Omega_+, \mathbb{R}^d)$ as $\rho(t) \to 0$,
(ii) $v(t) \in W^{1,p}(\Omega_- \cup \hat{M}(t) \cup \Omega_+, \mathbb{R}^d) \Rightarrow v^\rho(t) \in W^{1,p}(\Omega_- \cup (\hat{M}(t) + B_{\rho(t)}(0)) \cup \Omega_+, \mathbb{R}^d)$ with $B_\rho(0) \subset \mathbb{R}^d$,
(iii) $[\![v(t)]\!] \cdot \mathrm{n} \geq 0$ on $\Gamma_{\mathrm{C}} \Rightarrow [\![v^\rho(t)]\!] \cdot \mathrm{n} \geq 0$ on Γ_{C} .

We apply the construction of Proposition 5.2 to tailor a recovery sequence $(v_k)_k$ for any test function $v \in \mathfrak{U}_{z(t)}$. For our purpose, the radii $\rho = \rho(k,t)$ in Prop. 5.2 are given by

$$\rho(k,t) := \inf\{\rho > 0 \ : \ \operatorname{supp} z_k(t) \subset \operatorname{supp} z(t) + B_\rho(0)\}\,. \tag{5.12}$$

As proved in the forthcoming Propositions 6.6 and 6.9, we have $\rho(k,t) \to 0$ as $k \to \infty$ for all $t \in [0,T]$. Then, statement *(ii)* ensures that the sequence $(v_k)_k$, $v_k := v^{\rho(k,t)}$, does not jump on $\operatorname{supp} z_k(t)$ for a.a. $t \in (0,T)$. Moreover, $[\![v^{\rho(k,t)}]\!] \cdot \mathrm{n} \geq 0$ on Γ_{C} is given by statement *(iii)*. Statement *(i)* guarantees the desired convergence $v^{\rho(k,t)} \to v(t)$, only if $\rho(k,t) \to 0$ as $k \to \infty$. This is shown in Proposition 6.6 for $\operatorname{supp} z(t) = \emptyset$ and in Proposition 6.9 for $\operatorname{supp} z(t) \neq \emptyset$.

The above recovery sequence will now be used to state the Mosco-convergence of several functionals involved in the adhesive momentum balance.

Proposition 5.3 *Assume* (3.7c).
(1) Let $(z_k)_k \subset \mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0,1\})$ with $z_k \stackrel{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0,1\})$ as $k \to \infty$ and $X := W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}}; \mathbb{R}^d)$. Then, the functionals $\mathcal{J}_k(\cdot, z_k)$ (3.36) Mosco-converge in X as $k \to \infty$ to $\mathcal{J}_\infty(\cdot, z)$ (3.40).
(2) Let the assumptions of (1) hold. Then, the sequence $(\mathcal{F}_k(\cdot, z_k))_k$ (3.37) Mosco-converges in X as $k \to \infty$ to $\mathcal{F}_\infty(\cdot, z)$ (3.41).
(3) Let $X := L^p(0,T; W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}}; \mathbb{R}^d))$. For z_k, z satisfying (4.6c) consider the functionals

$$\widetilde{\mathcal{F}}_k(\cdot, z_k) : X \to [0,\infty], \qquad \widetilde{\mathcal{F}}_k(v, z_k) := \int_0^t \int_{\Omega\backslash\Gamma_{\mathrm{C}}} W_p(e(v(s)))\,\mathrm{d}x + \mathcal{F}_k(v(s), z_k(s))\,\mathrm{d}s\,, \tag{5.13a}$$

$$\widetilde{\mathcal{F}}_\infty(\cdot, z) : X \to [0,\infty], \qquad \widetilde{\mathcal{F}}_\infty(v, z) := \int_0^t \int_{\Omega\backslash\Gamma_{\mathrm{C}}} W_p(e(v(s)))\,\mathrm{d}x + \mathcal{F}_\infty(v(s), z(s))\,\mathrm{d}s\,. \tag{5.13b}$$

Then the sequence $(\widetilde{\mathcal{F}}_k(\cdot, z_k))_k$ Mosco-converges to the functional $\widetilde{\mathcal{F}}_\infty(\cdot, z)$ in X.

Proof: Ad *(1)*: The lim inf inequality (5.4) immediately follows from the fact that $\mathcal{J}_k(u_k, z_k) \geq 0$ for all $k \in \mathbb{N}$. This has to be combined with the observation that the limit pair (u,z) fulfills $z[\![u]\!] = 0$ on Γ_{C}, which can be checked arguing in the same way as throughout Step 0 of the proof of Thm. 5.1.

The lim sup condition (5.5) is proved by associating with each $v\in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ with $\mathcal{J}_\infty(v,z)<\infty$, i.e. $z[\![v]\!]=0$ on $\Gamma_{\rm C}$, the recovery sequence

$$v_k(x_1,y) := \begin{cases} v_{\rm sym}(x_1,y) + \xi^{{\rm supp}\,z}_{\rho(k)}(x_1,y)\, v_{\rm anti}(x_1,y) & \text{if } {\rm supp}\,z\neq\emptyset,\\ v(x,y) & \text{if } {\rm supp}\,z=\emptyset\,. \end{cases} \tag{5.14}$$

The construction for the case ${\rm supp}\,z=\emptyset$, is due to Proposition 6.6 stating that, if ${\rm supp}\,z=\emptyset$, then also ${\rm supp}\,z_k=\emptyset$ from a particular index k_0 on. For ${\rm supp}\,z\neq\emptyset$ the construction is the one from Proposition 5.2. The sequence $(v_k)_k$ strongly converges to v in X by *(i)* of Prop. 5.2. From *(ii)* and (5.12) it follows that $z_k[\![v_k]\!]=0$ for every $k\in\mathbb{N}$, hence $\mathcal{J}_k(v_k,z_k)=\mathcal{J}_\infty(v,z)=0$ and (5.5) is verified.

Clearly, ***(2)*** is an obvious consequence of *(1)*, also taking into account that, the construction of the recovery sequence $(v_k)_k$ preserves the non-penetration constraint, cf. *(iii)* in Prop. 5.2.

Ad *(3)*: Consider $v\in X=L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$. Again, the lim inf inequality (5.4) is easy to check. For a.a. $s\in(0,t)$ fixed a recovery sequence for $v(s)=v(s,x_1,y)$ is given by $v_k(s)=v_k(s,x_1,y)$ from (5.14). We prove that $v_k\to v$ strongly in X. Statement *(i)* of Prop. 5.2 yields that $v_k(s)\to v(s)$ strongly in $W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$, i.e. $\|v_k(s)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}\to\|v(s)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}$ pointwise a.e. in $(0,t)$. Moreover, due to $\xi^{\hat M}_{\rho(k)}(s,\cdot)\in[0,1]$ for a.a. $s\in(0,t)$, construction (5.14) gives $\|v_k(s)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}\le\|v(s)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}$ with $\|v(\cdot)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}\in L^p(0,t)$. Thus $\|v_k(\cdot)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}\to\|v(\cdot)\|_{W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)}$ in $L^p(0,t)$ due to the dominated convergence theorem. ■

Now, we want to carry out the limit passage in the momentum balance from adhesive to brittle exploiting convergences (4.6). As in Section 4.1, we observe that, there exists $\mu\in L^{p'}(0,T;L^{p'}(\Omega))$ such that, up to the extraction of a further (not relabeled) subsequence there holds

$${\rm DW}_p(e(u_k))\rightharpoonup\mu \quad \text{in } L^{p'}(0,T;L^{p'}(\Omega)). \tag{5.15}$$

Furthermore, a comparison in the reformulation (3.39) of the adhesive momentum equation for $(u_k,w_k,z_k)_k$ yields a bound for the sequence $(\lambda_k)_k\subset L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*)$ such that $\lambda_k(t)\in\partial_u\mathcal{F}_k(u_k(t),z_k(t))$ for almost all $t\in(0,T)$ and (u_k,w_k,z_k,λ_k) fulfill (3.39). Therefore, up to a subsequence,

$$\lambda_k\rightharpoonup\lambda \quad \text{in } L^{p'}(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*). \tag{5.16}$$

Convergences (5.15)–(5.16), combined with (4.6a)–(4.6h) allow us to show, as for (4.16), that the quintuple (u,w,z,μ,λ) for almost all $t\in(0,T)$ and all $v\in W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ fulfills

$$\int_{\Omega\backslash\Gamma_{\rm C}}\big({\rm DR}_2(e(\dot u(t)))+{\rm DW}_2(e(u(t)))-\mathbb{B}\Theta(w(t))+\mu(t)\big):e(v)\,{\rm d}x+\langle\lambda(t),v\rangle=\langle{\rm F}(t),v\rangle\,. \tag{5.17}$$

Thus, to be able to conclude that (5.17) is the momentum inclusion for the SBV-brittle limit, as in Section 4.1 we have to identify the limits

$$\mu(t)={\rm DW}_p(e(u(t))) \quad\text{and}\quad \lambda(t)\in\partial_u\mathcal{F}_\infty(u(t),z(t)) \quad \text{for a.a. } t\in(0,T). \tag{5.18}$$

For this, we exploit the MOSCO-convergence of the functionals $\widetilde{\mathcal{F}}_k(\cdot,z_k)$ defined in (5.13): indeed, we will apply the following Lemma 5.4 to the G-convergent sequence $(\partial_u\widetilde{\mathcal{F}}_k(\cdot,z_k))_k$.

Lemma 5.4 *Let X be a reflexive Banach space and $(A_k)_k$ a sequence of maximal monotone operators $A_k:X\rightrightarrows X^*$ which G-converge to a maximal monotone operator A. Then the following holds*

$$\left.\begin{array}{l}(u_k,u_k^*)\in X\times X^* \text{ with } u_k^*\in A_k(u_k),\\ u_k\rightharpoonup u \text{ in } X,\ u_k^*\rightharpoonup u^* \text{ in } X^*,\\ \limsup_{k\to\infty}\langle u_k^*,u_k\rangle_X\le\langle u^*,u\rangle_X\end{array}\right\}\quad\Rightarrow\quad (u,u^*)\in X\times X^* \text{ with } u^*\in A(u)\,. \tag{5.19}$$

The proof can be retrieved from the lines of the proof of [Att84, p. 361, Prop. 3.59].

We then obtain the following result on the limit passage in the momentum balance, where, as in Proposition 4.1, the identification (5.18) again implies the strong convergence of $(u_k)_k$ in $L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$.

Proposition 5.5 (Passage to the limit in the momentum equation as $k \to \infty$)
Assume (3.7), (3.8), (3.11), (3.13), *and let* $(u_k, w_k, z_k)_k$ *be a sequence of energetic solutions to the* SBV-*adhesive contact systems, for which convergences* (4.6a)–(4.6i) *to a limit triple* (u, w, z) *hold as* $k \to \infty$. *Then, the limit triple* (u, w, z) *satisfies the weak formulation* (3.28b) *of the momentum equation in the brittle case. In addition, there holds*

$$u_k \to u \text{ strongly in } L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)) \quad \text{and} \quad \Phi_k(u_k, z_k) \to \Phi_{\mathrm{b}}(u, z)\,. \tag{5.20}$$

Proof: To prove (5.18) we are going to show that $u^* \in X^*$ (with $X := L^p(0,T;W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$) given by $\langle u^*, v\rangle_X := \int_0^t \int_\Omega \mu(s)\colon e(v(s))\,\mathrm{d}x + \langle\lambda(s), v(s)\rangle\,\mathrm{d}s$ is such that $u^* \in \partial_u\widetilde{\mathcal{F}}_\infty(u, z)$. To this aim, we observe that the sequence $(u_k^*)_k \subset X^*$ defined by $\langle u_k^*, v\rangle_X := \int_0^t \int_\Omega \mathrm{DW}_p(e(u_k(s)))\colon e(v(s))\,\mathrm{d}x + \langle\lambda_k(s), v(s)\rangle\,\mathrm{d}s$ fulfills $u_k^* \in \partial_u\widetilde{\mathcal{F}}_k(u_k, z_k)$ and $u_k^* \rightharpoonup u^*$ in X^*. Then, we apply Lemma 5.4 to the sequence of maximal monotone subdifferential operators $(A_k)_k$ with $A_k := \partial_u\widetilde{\mathcal{F}}_k(\cdot, z_k) : X \rightrightarrows X^*$ and verify the lim sup-estimate in (5.19). For this, we again test (3.39) by u_k, integrate in time, and take the $\limsup_{k\to\infty}$. Thus, the very same calculations as throughout (4.19) give

$$\limsup_{k\to\infty} \int_0^t \left(\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mathrm{DW}_p(e(u_k)) : (e(u_k))\,\mathrm{d}x + \langle\lambda_k, u_k\rangle \right) \mathrm{d}s \leq \int_0^t \left(\int_{\Omega\backslash\Gamma_{\mathrm{C}}} \mu\colon e(u)\,\mathrm{d}x + \langle\lambda, u\rangle \right) \mathrm{d}s. \tag{5.21}$$

Hence, $u^* \in \partial_u\widetilde{\mathcal{F}}_\infty(u, z)$ and we conclude (5.18) as in the proof of Prop. 4.1.

For the convergence of the energies, i.e. $\Phi_k(u_k, z_k) \to \Phi_{\mathrm{b}}(u, z)$, it has to be shown that $\mathcal{J}_k(u_k, z_k) \to 0$. This is obtained by testing the adhesive momentum inequality (3.28a) by the recovery sequence $(v_k)_k$ constructed via (5.11) for the brittle limit solution u. Rearranging the terms in (3.28a) and exploiting that $z_k[\![u_k]\!] = 0$ a.e. on Γ_{C} by construction yields for a.a. $t \in (0, T)$ that

$$0 \leq \int_{\Gamma_{\mathrm{C}}} kz_k\big|[\![u_k]\!]\big|^2\,\mathrm{d}x \leq \int_{\Omega\backslash\Gamma_{\mathrm{C}}} \big(\mathrm{DR}_2(e(\dot{u}_k)) + \mathrm{DW}_2(e(u_k)) - \mathbb{B}\Theta(w_k) + \mathrm{DW}_p(e(u_k))\big)\colon e(v_k - u_k)\,\mathrm{d}x - \langle F, v_k - u_k\rangle$$
$$\longrightarrow 0 \qquad \text{as } k \to \infty,$$

since both $v_k \to u$ and $u_k \to u$ strongly in $W^{1,p}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ for a.a. $t \in (0, T)$. Hence $\int_{\Gamma_{\mathrm{C}}} kz_k\big|[\![u_k]\!]\big|^2\,\mathrm{d}x \to 0$ as $k \to \infty$ for a.a. $t \in (0, T)$. ■

5.2 Step 2: closedness of semistable sets

We now prove that the limit pair (u, z) complies with the semistability condition (3.29) by constructing a mutual recovery sequence, cf. Sec. 4.2, for the semistable sequence $(z_k)_k \subset L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}},\{0,1\}))$ fulfilling (4.6c). This construction is carried out in Proposition 5.7 below. It uses notation from the theory of BV-spaces, which can be found in Appendix B, cf. in particular Def. A.9, A.10. In order to guarantee that $\mathcal{R}_1(\tilde{z}_k - z_k) < \infty$ for the mutual recovery sequence $(\tilde{z}_k)_k$, we would like to apply a construction similar to the one developed in [TM10] for Sobolev-gradients, which mainly consists of considering the minimum of the stable sequence and the test function $\tilde{z}$. To deal with the gradient terms one exploits a chain rule formula for Sobolev-functions and the Lipschitz continuous minimum function, cf. [MM72]. A corresponding chain rule formula for distributional derivatives, see [ADM90], is more complicated to apply, as it also involves a kind of tangetial differential. For our purposes however, the following Theorem 5.2 on the decomposability of BV-functions, will provide an alternative construction that allows us to circumvent this general chain rule formula.

Theorem 5.2 ([AFP05, Th. 3.84] Decomposability of BV-functions)
Let $D \subset \mathbb{R}^m$. *Let* $\tilde{v}_1, \tilde{v}_2 \in BV(D)$ *and let* E *be a set of finite perimeter in* D, *with its reduced boundary* $\mathfrak{F}E$ *oriented by the generalized inner normal* ν_E. *Let* $v_{i\,\mathfrak{F}E}^{\pm}$ *denote the traces on* $\mathfrak{F}E \cap D$ *and* $\mathcal{X}_E$ *the characteristic function of the set* E. *Assume that* $\tilde{v}_{1\,\mathfrak{F}E}^{+}$ *and* $\tilde{v}_{2\,\mathfrak{F}E}^{-}$ *exist for* $\mathcal{H}^{m-1}$-*a.a.* $x \in \mathfrak{F}E \cap D$. *Then*

$$w := \tilde{v}_1\mathcal{X}_E + \tilde{v}_2\mathcal{X}_{D\backslash E} \in BV(D) \quad \text{if and only if} \quad \int_{\mathfrak{F}E\cap D} |\tilde{v}_{1\,\mathfrak{F}E}^{+} - \tilde{v}_{2\,\mathfrak{F}E}^{-}|\,\mathrm{d}\mathcal{H}^{m-1} < \infty\,. \tag{5.22}$$

If $w \in BV(D)$ the measure $\mathrm{D}w$ *is represented by*

$$\mathrm{D}w := \mathrm{D}\tilde{v}_1 \lfloor E^1 + \mathrm{D}\tilde{v}_2 \lfloor E^0 + (\tilde{v}^+_{1_{\mathfrak{F}E}} - \tilde{v}^-_{2_{\mathfrak{F}E}})\nu_E \otimes \mathcal{H}^{m-1} \lfloor (\mathfrak{F}E \cap D)\,, \tag{5.23}$$

where E^1 and E^0 denote the measure-theoretic interior and exterior of E, see Def. A.10.

We then have the following result (see [Tho11] for the proof).

Corollary 5.6 *Let $D \subset \mathbb{R}^m$ and $v \in BV(D)$ with $a \leq v \leq b$ $\mathcal{L}^m$-a.e. in D for constants $a, b \in \mathbb{R}$. Assume that Γ is a $\mathcal{H}^{m-1}$-rectifiable set oriented by ν. Then $a \leq v^{\pm}_{\Gamma}(x) \leq b$ for $\mathcal{H}^{m-1}$-a.a. $x \in D$.*

In the proof of the following result we will apply Thm. 5.2 and Cor. 5.6 with $D = \Gamma_{\mathrm{C}}$ and $m = d-1$.

Proposition 5.7 (Passage to the limit in the semistability condition as $k \to \infty$) *Assume* (3.8), (3.11), (3.13), *and let $(u_k, z_k)_k$ be a sequence of energetic solutions to the* SBV*-adhesive contact system, for which convergences* (4.6a)–(4.6f) *hold as $k \to \infty$. Then, the limit pair (u, z) fulfills the semistability condition* (3.29) *with the energy Φ_{b}.*

Proof: To prove (3.29) with Φ_{b}, it is sufficient to show for a.a. $t \in (0, T)$

$$\forall \tilde{z} \in \mathcal{Z}_{\mathrm{SBV}}: \qquad \Phi^{\mathrm{surf}}_{\mathrm{b}}\big(\llbracket u(t) \rrbracket, z(t)\big) \leq \Phi^{\mathrm{surf}}_{\mathrm{b}}\big(\llbracket u(t) \rrbracket, \tilde{z}\big) + \mathcal{R}_1(\tilde{z} - z(t)). \tag{5.24}$$

We will check (5.24) for $t \in (0, T)$ fixed, thus we will omit the variable t from now on. We verify the following MRS-condition: Let $(z_k)_k \subset \mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ be a semistable sequence for the energies $(\Phi_k)_k$, with $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$. Then, for all $\tilde{z} \in \mathcal{Z}$ there is a sequence $(\tilde{z}_k)_k \subset \mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ so that

$$\limsup_{k\to\infty} \big(\Phi_k\big(\llbracket u_k \rrbracket, \tilde{z}_k\big) - \Phi_k\big(\llbracket u_k \rrbracket, z_k\big) + \mathcal{R}_1(\tilde{z}_k - z_k)\big) \leq \Phi_{\mathrm{b}}\big(\llbracket u \rrbracket, \tilde{z}\big) - \Phi_{\mathrm{b}}\big(\llbracket u \rrbracket, z\big) + \mathcal{R}_1(\tilde{z} - z)\,. \tag{5.25}$$

In the proof of (5.25) we distinguish between the following two cases:

Case A: Let $\tilde{z} \in \mathcal{Z}$ be such that there exists a $\mathcal{L}^{d-1}$-measurable set $B \subset \Gamma_{\mathrm{C}}$ with $\mathcal{L}^{d-1}(B) > 0$ and $\tilde{z} > z$ on B. Then $\mathcal{R}(\tilde{z} - z) = \infty$ and (5.25) holds.

Case B: Let $\tilde{z} \leq z$ a.e. in Γ_{C}. Then, $\mathcal{R}(\tilde{z} - z) < \infty$. To avoid trivial cases, we also suppose that $\Phi_{\mathrm{b}}\big(\llbracket u \rrbracket, \tilde{z}\big) < \infty$, hence $0 \leq \tilde{z} \leq 1$ and $\tilde{z}\llbracket u \rrbracket = 0$ a.e. on Γ_{C}. To construct a mutual recovery sequence we set

$$\tilde{z}_k := \tilde{z}\mathcal{X}_{A_k} + z_k(1 - \mathcal{X}_{A_k}), \text{ where } A_k := \{x \in \Gamma_{\mathrm{C}} \,:\, 0 \leq \tilde{z}(x) \leq z_k(x)\} =: [0 \leq \tilde{z} \leq z_k]\,. \tag{5.26}$$

With this choice we ensure that $0 \leq \tilde{z}_k \leq z_k$ a.e. in Γ_{C}. Note that $\Gamma_{\mathrm{C}} \backslash A_k = [z_k < \tilde{z}] = [z_k = 0] \cap [\tilde{z} = 1]$. Since $z_k, \tilde{z} \in \mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ are the characteristic functions of sets Z_k, Z of uniformly bounded, finite perimeter, and relying on Prop. A.8, we find that

$$\exists\, C > 0 \;\; \forall\, k \in \mathbb{N}: \quad P(A_k, \Gamma_{\mathrm{C}}) = P(\Gamma_{\mathrm{C}} \backslash A_k, \Gamma_{\mathrm{C}}) \leq P(Z_k, \Gamma_{\mathrm{C}}) + P(Z, \Gamma_{\mathrm{C}}) \leq C.$$

Additionally, Cor. 5.6 implies that $|\tilde{z} - z_k| \leq 1$, $|\tilde{z}| \leq 1$ as well as $|z_k| \leq 1$ $\mathcal{H}^{d-1}$-a.e. on the respective reduced boundaries. Hence, Thm. 5.2 can be applied, yielding that $\tilde{z}_k \in \mathrm{BV}(\Gamma_{\mathrm{C}})$ for all $k \in \mathbb{N}$.

To verify that $\tilde{z}_k \to \tilde{z}$ in $L^1(\Gamma_{\mathrm{C}})$ we use that $[z_k < \tilde{z}] \subset [z_k < z] \subset [\varepsilon < |z_k - z|]$ for any $\varepsilon \in (0, 1)$, due to $z_k, z, \tilde{z} \in \mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0,1\})$. With Markov's inequality in estimate (M) we conclude

$$\mathcal{L}^{d-1}(\Gamma_{\mathrm{C}} \backslash A_k) \leq \mathcal{L}^{d-1}([\varepsilon < |z - z_k|]) \overset{\text{(M)}}{\leq} \varepsilon^{-1} \|z - z_k\|_{L^1(\Gamma_{\mathrm{C}})} \;\to 0\,, \tag{5.27}$$

the latter strong convergence due to $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0,1\})$. Thus, we infer

$$\|\tilde{z}_k - \tilde{z}\|_{L^1(\Gamma_{\mathrm{C}})} = \|z_k - \tilde{z}\|_{L^1(\Gamma_{\mathrm{C}} \backslash A_k)} \leq \mathcal{L}^{d-1}(\Gamma_{\mathrm{C}} \backslash A_k) \to 0\,. \tag{5.28}$$

In fact, since $(\tilde{z}_k)_k$ is bounded in $L^\infty(\Gamma_{\mathrm{C}})$ by construction, the above convergence improves to $\tilde{z}_k \to \tilde{z}$ in $L^q(\Gamma_{\mathrm{C}})$ for all $1 \leq q < \infty$. Using that $0 \leq \tilde{z}_k \leq z_k$ a.e. on Γ_{C}, we have that

$$\limsup_{k\to\infty} \frac{k}{2} \int_{\Gamma_{\mathrm{C}}} (\tilde{z}_k - z_k) |\llbracket u_k \rrbracket|^2 \, \mathrm{d}S \leq 0 = \int_{\Gamma_{\mathrm{C}}} \big(J_\infty(\llbracket u \rrbracket, \tilde{z}) - J_\infty(\llbracket u \rrbracket, z)\big) \, \mathrm{d}S.$$

Hence, in order to conclude the lim sup estimate (5.25), it remains to prove that

$$\limsup_{k\to\infty}\big(\mathcal{G}_{\rm b}(\tilde z_k)-\mathcal{G}_{\rm b}(z_k)+\mathcal{R}_1(\tilde z_k-z_k)\big)\le\limsup_{k\to\infty}\big(\mathcal{G}_{\rm b}(\tilde z_k)-\mathcal{G}_{\rm b}(z_k)\big)+\limsup_{k\to\infty}\mathcal{R}_1(\tilde z_k-z_k) \tag{5.29}$$

and we estimate the different terms in (5.29) separately.

Due to the strong L^1-convergence obtained in (5.28) and the fact that $\tilde z_k\le z_k$ for all $k\in\mathbb{N}$ by construction we conclude that $\mathcal{R}_1(\tilde z_k-z_k)\to\mathcal{R}_1(\tilde z-z)$ as $k\to\infty$.

Thus, to deduce the estimate for $\mathcal{G}_{\rm b}$, it remains to show that

$$\limsup_{k\to\infty}\big(|\mathrm{D}\tilde z_k|(\Gamma_{\rm C})-|\mathrm{D}z_k|(\Gamma_{\rm C})\big)\le|\mathrm{D}\tilde z|(\Gamma_{\rm C})-|\mathrm{D}z|(\Gamma_{\rm C})\,. \tag{5.30}$$

For this we recall that $\tilde z_k=\tilde z\mathcal{X}_{A_k}+z_k(1-\mathcal{X}_{A_k})$ as well as $z_k=z_k(\mathcal{X}_{A_k}+(1-\mathcal{X}_{A_k}))$ and we express their derivatives, i.e. the Radon measures $\mathrm{D}\tilde z_k$ and $\mathrm{D}z_k$, with the aid of formula (5.23):

$$|\mathrm{D}\tilde z_k|(\Gamma_{\rm C})=|\mathrm{D}\tilde z|(A_k^1)+|\mathrm{D}z_k|(A_k^0)+\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|\tilde z^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-2}\,, \tag{5.31}$$

where we applied Cor. 5.6 to determine the traces $\tilde z_k^\pm$ on the different parts of the reduced boundaries. Similarly we find

$$-|\mathrm{D}z_k|(\Gamma_{\rm C})=-\,|\mathrm{D}z_k|(A_k^1)-|\mathrm{D}z_k|(A_k^0)-\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-2}\,. \tag{5.32}$$

We note that both $|\mathrm{D}\tilde z_k|(A_k^0)=0$ and $-|\mathrm{D}z_k|(A_k^0)=0$ in (5.31). To establish (5.30) we have to show that $-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)\le-|\mathrm{D}z|(\Gamma_{\rm C})$ and that the boundary terms in (5.31) + (5.32) can be estimated as follows for all $k\in\mathbb{N}$:

$$\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|\tilde z^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-2}-\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-2}\le\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|\tilde z^+-\tilde z^-|\,\mathrm{d}\mathcal{H}^{d-2}. \tag{5.33}$$

To verify estimate (5.33) we use the information on the traces stated in Cor. 5.6 and distinguish between all possible relations. On $\mathfrak{F}A_k\cap\Gamma_{\rm C}$ it holds $0\le\tilde z^+\le z_k^+$ and $0\le z_k^-<\tilde z^-$ $\mathcal{H}^{d-2}$-a.e.. Hence, for $\mathcal{H}^{d-2}$-a.a. $x\in\mathfrak{F}A_k\cap\mathfrak{F}(\Gamma_{\rm C}\backslash A_k)\cap\Gamma_{\rm C}$ with

$$\begin{array}{lll}
z_k^+\le z_k^- & \text{it is} & \tilde z^+\le z_k^+\le z_k^-<\tilde z^-,\ \text{i.e.}\ |\tilde z^+-z_k^-|<|\tilde z^+-\tilde z^-|\,,\\
z_k^+>z_k^- & \text{it is either} & \tilde z^+\le z_k^-<z_k^+\le\tilde z^-,\ \text{i.e.}\ |\tilde z^+-z_k^-|\le|\tilde z^+-\tilde z^-|\,,\\
 & \text{or} & \tilde z^+\le z_k^-<\tilde z^-\le z_k^+,\ \text{i.e.}\ |\tilde z^+-z_k^-|\le|\tilde z^+-\tilde z^-|\,,\\
 & \text{or} & z_k^-<\tilde z^-\le\tilde z^+\le z_k^+,\ \text{i.e.}\ |\tilde z^+-z_k^-|\le|z_k^+-z_k^-|\,,\\
 & \text{or} & z_k^-<\tilde z^+\le z_k^+\le\tilde z^-,\ \text{i.e.}\ |\tilde z^+-z_k^-|\le|z_k^+-z_k^-|\,,\\
 & \text{or} & z_k^-<\tilde z^+<\tilde z^-\le z_k^+,\ \text{i.e.}\ |\tilde z^+-z_k^-|\le|z_k^+-z_k^-|\,.
\end{array}$$

Using these estimates and denoting the set of points, where one of the latter three relations holds by E, we find that

$$\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|\tilde z^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-2}-\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|z_k^+-z_k^-|\,\mathrm{d}\mathcal{H}^{d-2}\le\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}\backslash E}|\tilde z^+-\tilde z^-|\,\mathrm{d}\mathcal{H}^{d-2}-0\le\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|\tilde z^+-\tilde z^-|\,\mathrm{d}\mathcal{H}^{d-2}$$

Thus, (5.33) holds. In total we have obtained that the left-hand side of (5.30) can be estimated by

$$\begin{aligned}\limsup_{k\to\infty}\big(|\mathrm{D}\tilde z_k|(\Gamma_{\rm C})-|\mathrm{D}z_k|(\Gamma_{\rm C})\big)&\le\limsup_{k\to\infty}\Big(|\mathrm{D}\tilde z|(A_k^1)+\int_{\mathfrak{F}A_k\cap\Gamma_{\rm C}}|\tilde z^+-\tilde z^-|\,\mathrm{d}\mathcal{H}^{d-2}-|\mathrm{D}z_k|(A_k^1)\Big)\\&\le|\mathrm{D}\tilde z|(\Gamma_{\rm C})-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)\end{aligned} \tag{5.34}$$

To show that $-\liminf_{k\to\infty}|\mathrm{D}z_k|(A_k^1)\le-|\mathrm{D}z|(\Gamma_{\rm C})$ in (5.34) we first choose a subsequence $(z_k)_k$ such that the lim inf is attained. Then, we introduce the sets $U_n:=\bigcup_{k=n}^\infty(\Gamma_{\rm C}\backslash A_k)$. Since $\mathcal{L}^{d-1}(\Gamma_{\rm C}\backslash A_k)\to0$ as

$k \to \infty$ we may choose a further subsequence s.t. $\sum_{k=1}^{\infty} \mathcal{L}^{d-1}(\Gamma_C \backslash A_k) < \infty$. Hence for this subsequence, $\mathcal{L}^{d-1}(U_n) < \infty$ and $\mathcal{L}^{d-1}(U_n) \to 0$ as $n \to \infty$. We set $\lim_{n\to\infty} U_n = N$ and put $\Gamma_n := \Gamma_C \backslash U_n$, which satisfies $\Gamma_n \subset A_n$ for all $k \geq n$. Then, also $\Gamma_n^1 \subset A_k^1$ as well as $\Gamma_n^1 \subseteq \Gamma_{n+1}^1 \subset \Gamma_C^1$ for all $n \in \mathbb{N}$ by Cor. A.11, 2.). Since $\mathcal{L}^{d-1}(N) = 0$ we conclude that $(\Gamma_C \backslash N)^1 = \Gamma_C^1$ by Cor. A.11, 1.). This proves that $\Gamma_n^1 \to \Gamma_C^1$. Note that $\Gamma_C \subset \mathbb{R}^{d-1}$ is an open set, i.e. for all $x \in \Gamma_C$ there exists a constant $r_x > 0$ such that $B_r(x) \subset \Gamma_C$ for all $r \leq r_x$. Hence $\Gamma_C^1 = \Gamma_C$.

Keep $n \in \mathbb{N}$ fixed. Then the sets $\Gamma_n^1 \subset A_k^1$ can be used to find a set independent of $k \geq n$, so that the lower semicontinuity of the total variation functional can be exploited on Γ_n^1 for the sequence $z_k \overset{*}{\rightharpoonup} z$ in $\mathrm{SBV}(\Gamma_C, \{0,1\})$ and we have ensured that $\Gamma_n^1 \to \Gamma_C$. For all $k \geq n$ we have

$$-\liminf_{k\to\infty} |\mathrm{D}z_k|(A_k^1) \leq -\liminf_{k\to\infty} |\mathrm{D}z_k|(\Gamma_n^1) \leq -|\mathrm{D}z|(\Gamma_n^1) \ \to -|\mathrm{D}z|(\Gamma_C) \text{ as } n \to \infty\,.$$

This finishes the proof of estimate (5.34). Thus we conclude that the mutual recovery sequence $(\tilde{z}_k)_k$ given by (5.26) satisfies the lim sup-estimate (5.25). ∎

6 Support property of semistable sequences

We now investigate fine properties of the sequence $(z_k)_k$, which are exploited for proving the convergence of the momentum equation as $k \to \infty$ in Section 5.1. We will deduce such properties from the *sole* feature of semistability of the sequence $(z_k)_k$ with respect to the functionals $\Phi_k(u_k, \cdot)$.

The statement of the main result of this section, Thm. 6.1 below, is given for a generic sequence $(z_k) \subset L^\infty(0,T;\mathrm{SBV}(\Gamma_C;\{0,1\}))$ fulfilling the semistability condition (3.29). We refer to Remark 6.11 for further comments in this connection.

Theorem 6.1 (Support convergence) *Assume* (3.7c). *Let* $(z_k)_k \subset L^\infty(0,T;\mathrm{SBV}(\Gamma_C;\{0,1\}))$ *fulfill* (3.29) *for all* $k \in \mathbb{N}$. *Suppose that*

$$z_k(t) \overset{*}{\rightharpoonup} z(t) \qquad \textit{in } \mathrm{SBV}(\Gamma_C, \{0,1\}) \quad \textit{for all } t \in [0,T] \tag{6.1}$$

for some $z \in L^\infty(0,T;\mathrm{SBV}(\Gamma_C;\{0,1\}))$. *Set*

$$\rho(k,t) := \inf\{\rho > 0,\, \operatorname{supp} z_k(t) \subset \operatorname{supp} z(t) + B_\rho(0)\} \quad \textit{for all } t \in [0,T] \textit{ and all } k \in \mathbb{N}. \tag{6.2}$$

Then, for all $t \in [0,T]$ *we have support convergence, i.e.*

$$\operatorname{supp} z_k(t) \subset \operatorname{supp} z(t) + B_{\rho(k,t)}(0) \quad \textit{and} \quad \rho(k,t) \to 0 \quad \textit{as } k \to \infty. \tag{6.3}$$

Since the solutions $(z_k)_k$ of the thermal delamination problems satisfy the semistability (3.29) for all $t \in [0,T]$, hereafter in most of the arguments for proving Thm. 6.1 we will suppose $t \in (0,T)$ fixed and omit indicating the dependence of the functions and of the radii on t. Moreover, all the ensuing calculations only involve functions defined on the interface $\Gamma_C \subset \mathbb{R}^{d-1}$, hence we will use the abbreviation

$$m := d - 1\,.$$

The main idea we will develop is the following: Thanks to the SBV-gradient term in the energies $\Phi_k(u_k,\cdot)$ and $\Phi_b(u,\cdot)$ (cf. (3.18), (3.21)), the delamination parameters z_k, z in the adhesive and brittle SBV-models are characteristic functions $z_k, z \in \mathrm{SBV}(\Gamma_C, \{0,1\})$ of sets $Z_k, Z \subset \Gamma_C$ with finite perimeter. To be more precise, since the bulk energy is independent of z_k and since $\mathcal{J}_k(u_k, \tilde{z}) \leq \mathcal{J}_k(z_k, u_k)$ for all $\tilde{z} \leq z_k$, the semistability of z_k for $\Phi_k(u_k,\cdot)$, $k \in \mathbb{N} \cup \{\infty\}$, implies the semistability of the underlying set Z_k for the energy term $\mathcal{S}(\cdot) := \mathrm{b}P(\cdot, \Gamma_C) - a_0\mathcal{L}^m(\cdot)$, i.e.

$$\mathcal{S}(Z_k) \leq \mathcal{S}(\tilde{Z}) + \mathcal{R}_1(\tilde{z} - z_k) \quad \text{with} \quad \mathcal{S}(Z) := \mathrm{b}P(Z, \Gamma_C) - a_0\mathcal{L}^m(Z)\,. \tag{6.4}$$

In particular, arguing by contradiction to the semistability condition (6.4), we will be able to rule out the presence of isolated subsets or *thin cusps* of arbitrarily small $\mathcal{L}^m$-measure in Z_k, which would counteract property (6.3). To do so, we will resort to refined tools from geometric measure theory. We refer the reader for Appendix B for the related definitions and results.

Before starting with the proof of Thm. 6.1, we first motivate heuristically why isolated subsets or *thin cusps* in Z_k are *undesirable* properties of Z_k, if (6.3) has to be proved.

Preliminary considerations. First of all, one should be aware that elements $z \in \mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ (or in general $z \in L^1(\Gamma_{\mathrm{C}})$) are given by equivalence classes of functions differing on $\mathcal{L}^m$- null sets, only. Hence, in this setting, the support $\operatorname{supp} z$ and the null set N_z are rather defined similarly to the context of measures [Fed69, p. 60] by

$$\operatorname{supp} z := \cap\{A \,|\, A \text{ closed}, \mathcal{L}^m(Z\backslash A) = 0\} \text{ and } N_z := \Gamma_{\mathrm{C}}\backslash \operatorname{supp} z\,, \tag{6.5}$$

$$\text{where } Z := \{x \in \Gamma_{\mathrm{C}} \,|\, z(x) \neq 0\}\,. \tag{6.6}$$

This definition yields $\operatorname{supp} z$ closed and N_z open and for continuous functions it coincides with the conventional definition. As a direct consequence of (6.5) we have the following result.

Corollary 6.1 *Let* $z \in L^1(\Gamma_{\mathrm{C}})$. *Then* $\operatorname{supp} z + B_{\rho(k)}(0) \to \operatorname{supp} z$ *as* $\rho(k) \to 0$, *in the sense that* $\mathcal{L}^m(\operatorname{supp} z + B_{\rho(k)}(0)\backslash \operatorname{supp} z) \to 0$ *as* $\rho(k) \to 0$.

Proof: First assume that $\operatorname{supp} z = \emptyset$. Then $\emptyset + B_{\rho(k)}(0) = \emptyset$ so that the statement holds true. Now, assume that $x \in \operatorname{supp} z + B_{\rho(k)}(0)$ for all $\rho(k) > 0$. Then $x \notin N_z$, because N_z is an open set. Then, the thesis follows, observing that by monotonicity $(\mathcal{L}^m(N_z \cap \operatorname{supp} z + B_{\rho(k)}(0)))_k$ converges to $\mathcal{L}^m(N_z \cap \cap_{k\in\mathbb{N}}(\operatorname{supp} z + B_{\rho(k)}(0)))$ as $\rho(k) \to 0$. ■

While for every *fixed* $z \in L^1(\Gamma_{\mathrm{C}})$ we have $\operatorname{supp} z + B_{\rho(k)}(0) \to \operatorname{supp} z$ as $\rho(k) \to 0$, support convergence (6.3) is in general not true for arbitrary *sequences* $z_k \to z$ in $L^1(\Gamma_{\mathrm{C}})$ with $\operatorname{supp} z \neq \emptyset$. Clearly, for any sequence $z_k \to z$ in $L^1(\Gamma_{\mathrm{C}})$, which can attain values in the whole interval $[0,1]$, there is a sequence $(\rho(k))_k$ with $\rho(k) \geq 0$ such that $\operatorname{supp} z_k \subset \operatorname{supp} z + B_{\rho(k)}(0)$. This is due to the boundedness of Γ_{C}. But not necessarily $\rho(k) \to 0$ as $k \to \infty$, as can be seen from the following counterexample:

Example 6.2 Let $z = 1$ on a closed set $Z \subset \operatorname{int}\Gamma_{\mathrm{C}}$ and for all $k \in \mathbb{N}$ let $z_k = z$ on Z and $z_k = 1/k$ on $\Gamma_{\mathrm{C}}\backslash Z$. Then $z_k \to z$ uniformly on Γ_{C}. But for all $k \in \mathbb{N}$ we have $\operatorname{supp} z_k = \Gamma_{\mathrm{C}} \not\to \operatorname{supp} z = Z$ and hence, $\inf(\rho) = \rho(k) = \operatorname{dist}(\operatorname{supp} z, \partial\Gamma_{\mathrm{C}})$ for all $k \in \mathbb{N}$. Thus, $\operatorname{supp} z_k \subset \operatorname{supp} z + B_{\rho(k)}(0)$, but $\rho(k) \not\to 0$.

To exclude situations as above it is essential that $z_k(x) \in \{0,1\}$ a.e. on Γ_{C}, which is indeed given by the space $\mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$. Hence, z_k is the characteristic function of the finite-perimeter set Z_k as in (6.6).

However, working in $\mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ in general neither ensures

$$Z_k = \operatorname{supp} z_k \qquad \mathcal{L}^m\text{-a.e. on } \Gamma_{\mathrm{C}}, \tag{6.7}$$

nor support convergence (6.3). This can be seen from Example 6.3 below, which is constructed in the spirit of [Giu84, p. 24, Rem. 1.27] or [AFP05, p. 154, Ex. 3.53]. In fact, (6.7) and (6.3) will be deduced only by exploiting an additional qualification, namely (6.4).

Example 6.3 Let $Q := (0,1)^2$. The set of points with rational coordinates $Q \cap \mathbb{Q}^2$ is countable and can be arranged in a sequence $(q_j)_j$. For every $j \in \mathbb{N}$ and every $k \in \mathbb{N}$ we define the open ball $B(q_j, r_{jk})$ with radius $r_{jk} := 1/(4k \cdot 2^j)$ and center in q_j. Then, $\mathcal{L}^2(B(q_j, r_{jk})) = \pi/(16k^2 \cdot 2^{2j})$ and $P(B(q_j, r_{jk}), Q) = \pi/(2k \cdot 2^j)$. For all $k \in \mathbb{N}$ we set $Z_k := \cup_{j\in\mathbb{N}} B(q_j, r_{jk})$ and as $k \to \infty$ we obtain that

$$\mathcal{L}^2(Z_k) \leq \sum\nolimits_{j=1}^{\infty} \mathcal{L}^2(B(q_j, r_{jk})) = \pi/(8k^2) \to 0\,, \qquad P(Z_k, Q) \leq \sum\nolimits_{j=1}^{\infty} P(B(q_j, r_{jk}), \Gamma_{\mathrm{C}}) = \pi/k \to 0\,.$$

Hence $z_k \to z$ in $L^1(Q)$, where $Z = \emptyset$ (which can be identified with $Q \cap \mathbb{Q}^2$, in the sense that the respective indicator functions differ on a set of null Lebesgue measure). Additionally, the perimeters as well converge, since $P(Z_k, Q) \to 0 = P(Q \cap \mathbb{Q}^2, Q)$. Notice that, since $Q \cap \mathbb{Q}^2 \subset Z_k \subset Q$ the sets A_k are dense in Q for all $k \in \mathbb{N}$. Hence, by formula (6.5) we have $\operatorname{supp} z_k = Q$ for all $k \in \mathbb{N}$, whereas $\operatorname{supp} z = \emptyset$. This discrepancy is due to the fact that Z_k converge to a dense set of zero $\mathcal{L}^2$-measure, while $\mathcal{L}^2(Z_k) < \mathcal{L}^2(\operatorname{supp} z_k)$ because the topological boundary ∂Z_k of the sets Z_k is of positive $\mathcal{L}^2$-measure. Thus, $\operatorname{supp} z_k \not\to \operatorname{supp} z$. In particular, support convergence (6.3) does not hold, because $\emptyset + B_{\rho(k)}(0) = \emptyset$ and hence $\operatorname{supp} z_k \not\subset \emptyset$ for any $\rho(k) > 0$.

Examples 6.2 and 6.3 suggest that there two reasons for the failure of support convergence (6.3) under convergence (6.1):

$$1. \qquad \operatorname{supp} z = \emptyset \quad \text{and} \quad \operatorname{supp} z_k \neq \emptyset \;\text{ for all } k \in \mathbb{N}\,. \tag{6.8a}$$

$$2. \qquad \operatorname{supp} z \neq \emptyset \quad \text{and} \quad \rho(k) \not\to 0 \;\text{ for all } k \in \mathbb{N}\,. \tag{6.8b}$$

Because of $z_k \to z$ in $L^1(\Gamma_C)$ by (6.1), we observe that $z_k \to 0$ in $L^1(N_z)$. For the associated finite-perimeter sets Z_k as in (6.6) we have

$$\mathcal{L}^m(Z_k \cap N_z) \to 0 \qquad \text{as } k \to \infty. \tag{6.9}$$

Now, if the sets Z_k satisfy either (6.8a) or (6.8b) in combination with (6.9), then necessarily $Z_k \cap N_z$ turns into a set with arbitrarily small $\mathcal{L}^m$-measure, which is either isolated from $\operatorname{supp} z_k$, or is a *thin cusp.* These two properties are defined as follows:

Definition 6.4 (Isolated subset, thin cusp) *Let $A, B \subset \mathbb{R}^m$. Moreover, let A^1 denote the measure-theoretic interior of A (cf. Def. A.10), and let $\mathcal{X}_B$ be the characteristic function of B. We say that*

$$A \textit{ is isolated from } B \;\Leftrightarrow\; \exists\, \varepsilon_0 > 0\; \forall x \in A^1 : \; \operatorname{dist}(x, \operatorname{supp} \mathcal{X}_B) \geq \varepsilon_0\,, \tag{6.10}$$

$$A \textit{ is a } \text{thin cusp} \textit{ in } A \cup B \;\Leftrightarrow\; \begin{cases} \exists\, x \in A^1 : \; \operatorname{dist}(x, \operatorname{supp} \mathcal{X}_B) = 0 \textit{ and} \\ \exists\, \rho_0 > 0\; \exists\, A_k \subset A^1 \textit{ with } \mathcal{L}^m(A_k) > 0 : \\ \qquad\qquad \operatorname{dist}(A_k, \operatorname{supp} \mathcal{X}_B) > \rho_0\,. \end{cases} \tag{6.11}$$

As a direct consequence of Definition 6.4 we obtain for $B = \emptyset$:

$$\text{any set } A \subset \mathbb{R}^m \text{ with } A^1 \neq \emptyset \text{ is isolated from } B = \emptyset \text{ and it is not a thin cusp of } A \cup B. \tag{6.12}$$

Outline of the proof of Theorem 6.1. Let $(z_k)_k$, $z \subset \mathrm{SBV}(\Gamma_C; \{0,1\})$ be the characteristic functions of the sets $(Z_k)_k$, Z with finite perimeter, fulfilling the assumptions of Thm. 6.1, hence semistability (6.4).

Although $\mathcal{L}^m(Z_k \cap N_z) \to 0$, not necessarily $\mathcal{L}^m(\operatorname{supp} z_k \cap N_z) \to 0$, as we have learned from Example 6.3. However, exploiting (6.4) we will show that the sets Z_k (Z) neither contain isolated subsets nor thin cusps of arbitrarily small $\mathcal{L}^m$-measure. To be more precise:

1. in Lemma 6.5 we prove that characteristic functions z_k (z) of finite-perimeter sets Z_k (Z) which are isolated subsets of sufficiently small $\mathcal{L}^m$-measure violate (6.4), hence cannot be semistable for $\Phi_k(u_k, \cdot)$ $(\Phi_\infty(u, \cdot))$;
2. in Lemma 6.8 we prove that characteristic functions z_k (z) of finite-perimeter sets Z_k (Z) with sufficiently thin cusps violate (6.4), hence cannot be semistable for $\Phi_k(u_k, \cdot)$ $(\Phi_\infty(u, \cdot))$.

From Lemma 6.5 we will immediately deduce the support convergence (6.3) in the case $\operatorname{supp} z = \emptyset$, cf. Prop. 6.6. Relying on both Lemma 6.5 and Lemma 6.8, we will prove (6.3) if $\operatorname{supp} z \neq \emptyset$ in Prop. 6.9.

The main tools for this are the isoperimetric inequality (in the case of isolated sets) and the relative isoperimetric inequality (in the case of thin cusps), which we collect in the result below.

Theorem 6.2 (Isoperimetric inequalities [AFP05, Th. 3.46 and (3.43)]) *Let $m > 1$ be an integer. Let A be a set of finite perimeter in $\mathbb{R}^m$.*

1. *Isoperimetric inequality: For any set A of finite perimeter in $\mathbb{R}^m$ either A or $\mathbb{R}^m \backslash A$ have finite Lebesgue measure and*

$$\min\left\{\mathcal{L}^m(A), \mathcal{L}^m(\mathbb{R}^m \backslash A)\right\}^{\frac{m-1}{m}} \leq c_m P(A, \mathbb{R}^m) \tag{6.13}$$

   for some dimensional constant c_m.

2. *Relative isoperimetric inequality in balls: For each ball $B_\rho(y) \subset \mathbb{R}^m$ it holds*

$$\min\left\{\mathcal{L}^m(A \cap B_\rho(y)), \mathcal{L}^m(B_\rho(y) \backslash A)\right\}^{\frac{m-1}{m}} \leq \tilde{c}_m P(A, B_\rho(y)) \tag{6.14}$$

   for some dimensional constant $\tilde{c}_m$.

We will first treat the case of isolated subsets and then solve the more general and more problematic case of thin cusps.

Elimination of isolated subsets and support convergence (6.3) if $\mathrm{supp}(z(t)) = \emptyset$. In order to prove that characteristic functions z_k (z) of sets Z_k (Z) which contain isolated subsets of sufficiently small $\mathcal{L}^m$-measure cannot be semistable for $\mathcal{S}$ in (6.4), we have to construct a particular test function for which (6.4) is violated. Since we deal with an isolated subset we can obtain such a test function simply by eliminating this subset.

Lemma 6.5 (Elimination of isolated subsets by semistability) *Let $k \in \mathbb{N} \cup \{\infty\}$ fixed. Assume* (3.7c). *Let z_k be the characteristic function of a $\mathcal{L}^m$-measurable set Z_k with finite perimeter, semistable for $\Phi_k(u_k, \cdot)$, hence* (6.4). *Then, the measure-theoretic interior Z_k^1 does not contain any subset A_k isolated from $Z_k \backslash A_k$ with $0 < \mathcal{L}^m(A_k) < \mathrm{b}^m\big((a_0{+}a_1)c_m\big)^{-m}$, where c_m is from the isoperimetric inequality* (6.13).

Proof: By contradiction, assume that Z_k^1 contains a subset A_k isolated from $Z_k \backslash A_k$ with $0 < \mathcal{L}^m(A_k) < \mathrm{b}^m\big((a_0{+}a_1)c_m\big)^{-m}$. We test (6.4) with the characteristic function $\tilde{z}_k$ of the set $\widetilde{Z}_k := Z_k \backslash A_k$, which clearly satisfies $\tilde{z}_k \le z_k$ a.e. in Γ_{C} and hence $\mathcal{R}_1(\tilde{z}_k - z_k) < \infty$. As A_k is isolated, (6.4) yields

$$\mathrm{b}P(A_k, \Gamma_{\mathrm{C}}) \le (a_0{+}a_1)\mathcal{L}^m(A_k) \le (a_0{+}a_1)(c_m P(A_k, \Gamma_{\mathrm{C}}))\mathcal{L}^m(A_k)^{1/m} \,. \tag{6.15}$$

The second estimate is due to the isoperimetric inequality (6.13) using that $\min\{\mathcal{L}^m(A_k), \mathcal{L}^m(\mathbb{R}^m \backslash A_k)\} = \mathcal{L}^m(A_k)$. Since $\mathcal{L}^m(A_k) > 0$, we have $P(A_k, \Gamma_{\mathrm{C}}) > 0$. Thus, we can divide by $P(A_k, \Gamma_{\mathrm{C}})$ in (6.15), which yields $\mathrm{b}\big((a_0 + a_1)c_m\big)^{-1} \le \mathcal{L}^m(A_k)^{1/m}$. This is in contradiction to the assumption $\mathcal{L}^m(A_k) < \mathrm{b}^m\big((a_0{+}a_1)c_m\big)^{-m}$. Hence, the characteristic function z_k of the set Z_k containing the isolated subset A_k with $0 < \mathcal{L}^m(A_k) < \mathrm{b}^m\big((a_0{+}a_1)c_m\big)^{-m}$ cannot be semistable, neither for $\mathcal{S}$ nor for $\Phi_k(u_k, \cdot)$. ∎

As a direct consequence of Lemma 6.5 and (6.12), we exclude case (6.8a) for semistable sequences.

Proposition 6.6 (Support convergence of semistable sequences if $\mathrm{supp}\, z = \emptyset$) *Assume* (3.7c). *Let $(z_k)_k$, $z \subset L^\infty(0, T; \mathrm{SBV}(\Gamma_{\mathrm{C}}; \{0,1\}))$ be as in Thm. 6.1. Assume that $\mathrm{supp}\, z(t) = \emptyset$ at some $t \in (0, T)$. Then, there is an index $k_0(t) \in \mathbb{N}$ such that also $\mathrm{supp}\, z_k(t) = \emptyset$ for all $k \ge k_0(t)$.*

Proof: Since $\mathrm{supp}\, z(t) = \emptyset$ and $z_k(t) \overset{*}{\rightharpoonup} z(t)$ in $\mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$, we have $z_k \to 0$ in $L^1(\Gamma_{\mathrm{C}})$. Now, in view of relation (6.12), the sets Z_k are isolated from $\mathrm{supp}\, z$ with $\mathcal{L}^m(Z_k) \to 0$. In particular, there is an index $k_0 \in \mathbb{N}$ such that $\mathcal{L}^m(Z_k) < \mathrm{b}^m\big((a_0{+}a_1)c_m\big)^{-m}$ for all $k \ge k_0$. By Lemma 6.5, this is in contradiction to the semistability of z_k for Φ_k. Thus, we conclude that $z_k \equiv 0$ for all $k \ge k_0$. ∎

Elimination of thin cusps and support convergence (6.3) if $\mathrm{supp}(z(t)) \neq \emptyset$. Our remaining task is to verify (6.3) for $\mathrm{supp}\, z(t) \neq \emptyset$. From now on, we will again omit to indicate the time variable t. Let us proceed by contradiction and assume that (6.3) does not hold. We will show that this leads to a contradiction to the semistability of z_k for $\Phi_k(u_k, \cdot)$. The sequence $(\rho(k))_k$ is uniformly bounded by the boundedness of Γ_{C}. Hence, we can find a subsequence converging to the lim sup of the whole sequence $\limsup_{k\to\infty} \rho(k) =: \rho_0$. Moreover, due to $z_k \to z$ in $L^1(\Gamma_{\mathrm{C}})$, there is a further subsequence which converges pointwise a.e. on Γ_{C}. Hence, to carry out the contradiction argument, we can restrict ourselves to a (not relabeled) subsequence of weak solutions which satisfies

- z_k is the characteristic function of a set Z_k with finite perimeter.
- $(z_k)_k$ realizes $\rho_0 := \limsup_{k\to\infty} \rho(k)$ in (6.3).
- $z_k \to z$ pointwise $\mathcal{L}^m$-a.e. on Γ_{C} .

(6.16)

For this subsequence the contradiction to (6.3) reads

$$\mathrm{supp}\, z_k \subset \mathrm{supp}\, z + B_{\rho(k)}(0) \text{ for all } k \in \mathbb{N} \quad \text{and} \quad \rho(k) \to \rho_0 > 0 \text{ as } k \to \infty \,. \tag{6.17}$$

Because of $\rho_0 > 0$ we can find a further subsequence and an index k_0 such that

$$\rho(k) > \rho_0/2 \quad \text{for all } k \ge k_0 \,. \tag{6.18}$$

Assume that $\mathcal{L}^m(\operatorname{supp} z_k \cap N_z) > 0$ (otherwise $\rho(k) = 0$). Then, also $\mathcal{L}^m(Z_k \cap N_z) > 0$. Thus, for every $k \geq k_0$ there is a point $y_k \in Z_k^1 \cap N_z$ with the property

$$\operatorname{dist}(y_k, \operatorname{supp} z) \geq \rho_0/2 \quad \text{and thus} \quad B_{\rho_0/4}(y_k) \subset N_z\,. \tag{6.19}$$

For shorter notation we set

$$\forall k \geq k_0: \quad A_k := Z_k \cap B_{\rho_0/4}(y_k)\,. \tag{6.20}$$

The convergence $z_k \to z$ pointwise $\mathcal{L}^m$-a.e. in particular implies that for every $\varepsilon \in (0,1)$ and $\mathcal{L}^m$-a.a. $y \in N_z$ there is an index $k(y,\varepsilon)$ such that for all $k \geq k(y,\varepsilon)$ it holds $|z_k(y)| = |z_k(y) - z(y)| < \varepsilon$, hence

$$z_k \overset{*}{\rightharpoonup} z \text{ in } \mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\}) \,\&\, (6.16) \;\;\Rightarrow\;\; \mathcal{L}^m(A_k) \leq \mathcal{L}^m(Z_k \cap N_z) \to 0 \;\text{ as }\; k_0 \leq k \to \infty\,. \tag{6.21}$$

Therefore, if additionally (6.19) must be satisfied, the sets $Z_k \cap N_z$ either have to turn into isolated sets or into thin cusps. The first case is already ruled out by Lemma 6.5.

The occurrence of thin cusps will be excluded in Lemma 6.8 by showing that the semistability (6.4) of a set Z for $\mathcal{S}$ (resp. its characteristic function z), is violated for a particular test function $\tilde{z}$ being the characteristic function of a suitable set $\widetilde{Z}$. Since we are working in $\mathrm{SBV}(\Gamma_{\mathrm{C}}, \{0,1\})$ this test function $\tilde{z}$ can be constructed by "cutting off the thin cusp" in a suitable way. More detailed, this cut-off yields that $\widetilde{Z}^1 \subset Z^1$. Additionally, it generates a new surface and we have to ensure that this new surface is smaller than the surface of the part which is cut off. For this, we consider

$$y \in Z^1\,, \quad \rho_y^* := \operatorname{dist}(y, \partial\Gamma_{\mathrm{C}}) \;\text{ and }\; A(y) := Z^1 \cap B_{\rho_y^*}(y)\,, \tag{6.22}$$

with Z^1 the measure-theoretic interior of Z, cf. Def. A.10; later on, in Prop. 6.9, to prove support convergence, we will choose $y = y_k$ and $\rho_y^* = \rho_0/4$. To show the existence of a suitable cut-off for a thin cusp $A(y)$ of sufficiently small $\mathcal{L}^m$-measure we fix $\alpha \in (0,1)$ and show

$$0 < \mathcal{L}^m(A(y)) < M(\rho_y^*) \;\;\Rightarrow\;\; \exists\, \rho \in [\rho_y^*/2, \rho_y^*]:\; 0 < \mathcal{H}^{m-1}(A(y) \cap \partial B_\rho(y)) < \alpha P(A(y), B_\rho(y))\,, \tag{6.23a}$$

$$\text{where } M(\rho_y^*) := \min\{(\rho_y^*/2)^m \mathcal{L}^m(B_1(0))/2, (\alpha\rho_y^*)^m/(2\tilde{c}_m m)^m, \mathrm{b}^m(1-\alpha)^m/(\tilde{c}_m(a_0+a_1))^m\}\,. \tag{6.23b}$$

In order to verify implication (6.23a) we assume the contrary, i.e.

$$\mathcal{L}^m(A(y)) < M(\rho_y^*) \text{ and } \forall \rho \in [\rho_y^*/2, \rho_y^*]:\; \mathcal{H}^{m-1}(A(y) \cap \partial B_\rho(y)) \geq \alpha P(A(y), B_\rho(y))\,. \tag{6.24}$$

For the contradiction argument we will use the volume formula, cf. [EG92, p. 197]

$$y \in A \subset \mathbb{R}^m,\, A \subset B_R(y) \quad\Rightarrow\quad \forall\, R > 0:\;\; \mathcal{L}^m(A) = \int_0^R \mathcal{H}^{m-1}(A^1 \cap \partial B_\rho(y))\,\mathrm{d}\rho \tag{6.25}$$

and we will exploit the relative isoperimetric inequality in balls (6.14).

Lemma 6.7 *Assume* (3.7c)*. Let $A(y)$ be given by* (6.22)*. Then implication* (6.23a) *is true.*

Proof: We assume that (6.23a) is false, i.e. we have (6.24), instead. By (6.23b) it is $\mathcal{L}^m(A(y)) = \mathcal{L}^m(Z^1 \cap B_{\rho_y^*}(y)) < M(\rho_y^*) \leq \mathcal{L}^m(B_{\rho_y^*/2}(y))/2$ and hence we have

$$\forall \rho \in [\rho_y^*/2, \rho_y^*]:\; \min\{\mathcal{L}^m(A(y) \cap B_\rho(y)), \mathcal{L}^m(B_\rho(y) \backslash A(y))\} = \mathcal{L}^m(A(y) \cap B_\rho(y))\,. \tag{6.26}$$

Moreover, applying the relative isoperimetric inequality (6.14) on the estimate in (6.24) yields that for all $\rho \in [\rho_y^*/2, \rho_y^*]$ it is

$$\mathcal{H}^{m-1}(A(y) \cap \partial B_\rho(y)) \geq \frac{\alpha}{\tilde{c}_m} \mathcal{L}^m(A(y) \cap B_\rho(y))^{\frac{m-1}{m}}\,, \tag{6.27}$$

where $\mathcal{H}^{m-1}(A(y)\cap\partial B_\rho(y)) = \frac{\mathrm{d}}{\mathrm{d}\rho}\mathcal{L}^m(A(y)\cap B_\rho(y))$ by (6.25). Since $\mathcal{L}^m(A(y)\cap B_\rho(y)) > 0$ for all $\rho\in[\rho_y^*/2,\rho_y^*]$, we can divide by $\mathcal{L}^m(A(y)\cap B_\rho(y))^{\frac{m-1}{m}}$ in (6.27). Integration over $\rho\in(\rho_y^*/2,\rho_y^*)$ with the substitution $u=\mathcal{L}^m(A(y)\cap B_\rho(y))$, $\mathrm{d}u=\mathcal{H}^{m-1}(A(y)\cap\partial B_\rho(y))\,\mathrm{d}\rho$ then yields

$$\begin{aligned} I := \int_{\rho_y^*/2}^{\rho_y^*} \mathcal{L}^m(A(y)\cap B_\rho(y))^{\frac{1-m}{m}}\mathcal{H}^{m-1}(A(y)\cap\partial B_\rho(y))\,\mathrm{d}\rho = \int_a^b u^{\frac{1-m}{m}}\,\mathrm{d}u = \big[mu^{\frac{1}{m}}\big]_a^b \\ = m\big(\mathcal{L}^m(A(y)\cap B_{\rho_y^*}(y))^{\frac{1}{m}} - \mathcal{L}^m(A(y)\cap B_{\rho_y^*/2}(y))^{\frac{1}{m}}\big)\,, \end{aligned}$$

where we have used $a=\mathcal{L}^m(A(y)\cap B_{\rho_y^*/2}(y))$ and $b=\mathcal{L}^m(A(y)\cap B_{\rho_y^*}(y))$ for shorter notation. Altogether, (6.27) then implies $I\geq\alpha\tilde{c}_m^{-1}\int_{\rho_y^*/2}^{\rho_y^*}\mathrm{d}\rho=\alpha\rho_y^*/(2\tilde{c}_m)$. In view of (6.23b) this leads to the contradiction

$$0<\frac{\alpha\rho_y^*}{2\tilde{c}_m m}\leq\big(\mathcal{L}^m(A(y)\cap B_{\rho_y^*}(y))^{\frac{1}{m}} - \mathcal{L}^m(A(y)\cap B_{\rho_y^*/2}(y))^{\frac{1}{m}}\big)<\frac{\alpha\rho_y^*}{2\tilde{c}_m m}\,.$$

We conclude that (6.24) is false, thus (6.23a) holds true. ■

Properties (6.23) will now be used in the proof of Lemma 6.8.

Lemma 6.8 (Elimination of thin cusps by semistability) *Assume* (3.7c). *Let* $k\in\mathbb{N}\cup\{\infty\}$ *and let* $\tilde{c}_m$ *be the constant from the relative isoperimetric inequality* (6.14). *Assume that* z *is semistable for* $\Phi_k(u_k,\cdot)$, *hence* (6.4) *holds. Then* Z^1 *does not contain any thin cusp* $A(y)$ *with the properties* (6.23).

Proof: Suppose that Z^1 contains a thin cusp $A(y)$ with the properties (6.23). We test (6.4) with the characteristic function $\tilde{z}$ of the set $\widetilde{Z}:=Z^1\backslash(A(y)\cap B_\rho(y))$, with $\rho\in[\rho_y^*/2,\rho_y^*]$ and $\alpha\in(0,1)$ such that estimate (6.23a) holds. For $\widetilde{Z}$, resp. $\tilde{z}$, semistability (6.4) holds. In particular, the above construction of $\tilde{z}$ ensures that $\tilde{z}\leq z$, so that $\mathcal{R}_1(\tilde{z}-z)=\int_{\Gamma_{\mathrm{C}}}a_1(z-\tilde{z})\,\mathrm{d}S$. Moreover, in view of Def. A.9, A.13 and (A.43), it yields $\mathrm{J}_{\tilde{z}}=\mathfrak{F}Z^1\backslash(\mathfrak{F}A(y)\cap B_\rho(y))\cup(A(y)\cup\partial B_\rho(y))$. Hence, (6.4) leads to the following relation

$$\mathrm{b}P(A(y),B_\varrho(y))\leq(a_0+a_1)\mathcal{L}^m(A(y)\cap B_\rho(y))+\mathrm{b}\mathcal{H}^{m-1}(A(y)\cap\partial B_\rho(y))\,. \tag{6.28}$$

Property (6.23b) via the isoperimetric inequality (6.14) implies that $P(A(y),B_\rho(y))>0$, since it ensures

$$\mathcal{L}^m(A(y)\cap B_\rho(y))=\min\{\mathcal{L}^m(A(y)\cap B_\rho(y)),\mathcal{L}^m(B_\rho(y)\backslash A(y))\}\,. \tag{6.29}$$

Hence, (6.28) is equivalent to

$$\mathrm{b}\leq(a_0+a_1)\frac{\mathcal{L}^m(A(y)\cap B_\rho(y))}{P(A(y),B_\rho(y))}+\mathrm{b}\,\frac{\mathcal{H}^{m-1}(A(y)\cap\partial B_\rho(y))}{P(A(y),B_\rho(y))}\,, \tag{6.30}$$

where $\mathcal{H}^{m-1}(A(y)\cap\partial B_\rho(y))/P(A(y),B_\rho(y))<\alpha$ by (6.23a). Additionally, the relative isoperimetric inequality in balls (6.14) together with (6.29) and (6.23b) yields

$$(a_0+a_1)\,\frac{\mathcal{L}^m(A(y)\cap B_\rho(y))}{P(A(y),B_\rho(y))}\leq(a_0+a_1)\tilde{c}_m\mathcal{L}^m(A(y)\cap B_\rho(y))^{\frac{1}{m}}<\mathrm{b}(1-\alpha)\,. \tag{6.31}$$

Inserting these estimates into (6.30) then generates the contradiction $1<1$, which concludes the proof. ■

Now we are in a position to show the support convergence of semistable sequences (6.3). For this, we assume its contrary (6.17) and show that this setting leads to arbitrarily thin cusps in the supports, which contradict semistability according to Lemma 6.8.

Proposition 6.9 (Support convergence for semistable sequences if supp $z\neq\emptyset$) *Assume* (3.7c). *Let* (z_k), $z\subset L^\infty(0,T;\mathrm{SBV}(\Gamma_{\mathrm{C}};\{0,1\}))$ *be as in Thm. 6.1. Then the support convergence* (6.3) *holds true.*

Proof: Fix $t\in(0,T)$ outside a set of null Lebesgue measure, such that convergence (6.1) and semistability (6.4) hold for $(z_k(t))_k$. For shorter notation we omit to indicate the time-dependence of z_k and $\rho(k)$. Suppose by contradiction that (6.16) holds. Then, we can proceed as in (6.18) and (6.19) to obtain the sets $A_k = Z_k \cap N_z$ from (6.20). In (6.21) it is assured that these sets A_k converge to $\mathcal{L}^m$-zero sets and hence turn into small isolated subsets and arbitrarily thin cusps. Lemma 6.7 then guarantees the properties (6.23), since, clearly, all the arguments can be repeated for $\rho^*_{y_k} = \rho_0/4$. But, Lemmata 6.5 and 6.8 state that characteristic functions z_k of sets Z_k, which contain sufficiently small isolated sets and sufficiently thin cusps A_k, cannot be semistable, hence the contradiction to $\rho_0 > 0$. This implies that $\rho(k) \to 0$ for the *whole* sequence $(\rho(k))_k$ and hence support convergence holds for the *whole* sequence $(z_k)_k$. ■

Remark 6.10 We note that Proposition 6.9 only requires the semistability of the delamination variables $(z_k)_k$ of the SBV-adhesive contact systems: The semistability of the limit function z is not required. Nonetheless, the proof of the semistability for the limit function is completely independent from the support convergence property (6.3).

Remark 6.11 (Generality) The results of Lemmata 6.5, 6.8 and Propositions 6.6, 6.9 are solely based on semistability and strong L^1-convergence. In other words, further properties of the delamination models such as temperature dependence or visco-elasticity have no influence. In particular, Propositions 6.6 and 6.9 also hold for energetic solutions in the fully rate-independent setting, where solutions are characterized as satisfying an energy balance and a *global stability* condition, see (1.6).

Remark 6.12 (Open problem: from brittle SBV-delamination to Griffith-type delamination) It is an open problem to get rid of the SBV-gradient regularization, like in [MRT12] in the limit passage from Sobolev gradient to Griffith-type delamination. In the present context, this would mean passing to zero with the coefficient b in the gradient term $\mathcal{G}_\mathrm{b}(z) := \mathrm{b}P(Z,\Gamma_\mathrm{C})$ contributing to the energy Φ_b.

Seemingly, the main difficulty attached to the limit passage $\mathrm{b} \to 0$ is the proof of the support convergence (5.3), which in turn would be crucial for passing to the limit in the momentum equation in this case as well. More specifically, we highlight that Lemmata 6.5 and 6.8 exclude the presence of isolated subsets A_k with $\mathcal{L}^m(A_k) < \mathrm{b}\big((a_0 + a_1)c_m\big)^{-m}$ and of thin cups A_k fulfilling (6.23b), respectively. Both constraints on the measure of A_k explicitly involve the constant $\mathrm{b} > 0$. Therefore, the passage from SBV-brittle delamination to Griffith-type delamination as $\mathrm{b} \to 0$ would bring along a loss of uniform boundedness in $\mathrm{SBV}(\Gamma_\mathrm{C})$ for the sequence $(z_\mathrm{b})_\mathrm{b}$ of delamination variables for the SBV-brittle delamination systems. Indeed, only the uniform bound in $L^\infty(\Gamma_\mathrm{C})$ would remain. Hence, the limit of a semistable sequence $(z_\mathrm{b})_\mathrm{b} \subset \mathrm{SBV}(\Gamma_\mathrm{C}, \{0,1\})$ with $z_\mathrm{b} \overset{*}{\rightharpoonup} z$ in $L^\infty(\Gamma_\mathrm{C})$ would be an L^∞-function, only, which can of course contain arbitrarily small isolated subsets or thin cusps. Indeed, for $\mathrm{b} \to 0$, the uniform bound on the measure of isolated subsets and thin cusps is lost and the larger the perimeters of the approximating functions may get, the smaller the undesirable subsets in their supports may be.

A Appendix

A.1 Time-discretization for the Modica-Mortola adhesive system

In this section we outline the proof of Theorem 4.2. We perform a semi-implicit time-discretization: for a given time-step $\tau > 0$, we consider the equidistant partition $\{t^0_\tau = 0 < \ldots < t^j_\tau = j\tau < \ldots < t^{J_\tau}_\tau = T\}$ of $[0,T]$. We approximate the data F, f by local means, i.e. setting $F^j_\tau := \frac{1}{\tau}\int_{t^{j-1}_\tau}^{t^j_\tau} F(s)\mathrm{d}s$ and $f^j_\tau := \frac{1}{\tau}\int_{t^{j-1}_\tau}^{t^j_\tau} f(s)\mathrm{d}s$ for all $j = 1, \ldots, J_\tau$. Then, from F^j_τ and f^j_τ we define $\mathrm{F}^j_\tau \in W^{1,p}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)^*$ as in (3.12). Furthermore, for technical reasons related to the existence proof of Problem A.1 below, we need to approximate H and h by means of suitably constructed discrete data $\{H^j_\tau\}_{j=1}^{J_\tau}$, $\{h^j_\tau\}_{j=1}^{J_\tau}$ with

$$H^j_\tau \in W^{1,2}(\Omega)^*, \qquad h^j_\tau \in H^{1/2}(\partial\Omega)^* \qquad \text{for all } j = 1, \ldots, J_\tau, \tag{A.1}$$

and analogously define $\mathrm{H}^j_\tau \in W^{1,2}(\Omega)^*$ as in (3.12). Finally, we approximate the initial datum u_0 with a sequence $\{u_{0,\tau}\} \subset W^{1,\gamma}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)$ (with $\gamma > \max\{p, \frac{2\omega}{\omega-1}\}$, see Problem A.1) such that

$$\lim_{\tau\downarrow 0} \sqrt[\gamma]{\tau}\|e(u_{0,\tau})\|_{L^\gamma(\Omega;\mathbb{R}^d)} = 0, \qquad u_{0,\tau} \to u_0 \qquad \text{in } W^{1,p}(\Omega;\mathbb{R}^d) \;\text{ as }\; \tau \to 0. \tag{A.2}$$

We consider the following time-discrete approximation of the Modica-Mortola adhesive system. Therein, we add to the momentum equation the regularizing term $\tau|e(u)|^{\gamma-2}e(u)$, with $\gamma > \max\{p, \frac{2\omega}{\omega-1}\}$ and $\omega > \frac{2d}{d+2}$ as in (3.8b): this enables us to apply to system (A.4) and (A.5) existence results from the theory of pseudo-monotone operators, see the proof of Lemma A.3. Equations (A.4) and (A.5) are coupled with the time-incremental minimization problem (A.6), whose solutions in particular fulfill the *discrete* flow rule (A.7). However, (A.6) contains more information than (A.7). It will enable us to prove the *discrete* mechanical energy inequality (A.10) and semistability (A.11) in Lemma A.5, which in turn play a crucial role in the proof of the a priori estimates of Prop. A.6. For further comments on Problem A.1, we refer to Remark A.2 below.

Problem A.1 Let $\gamma := \max\{p, \frac{2\omega}{\omega-1}\}$. Given

$$u_\tau^0 = u_{0,\tau}, \qquad u_\tau^{-1} = u_{0,\tau} - \tau\dot{u}_0, \qquad z_\tau^0 = z_0, \qquad w_\tau^0 = w_0, \tag{A.3}$$

find $\{(u_\tau^j, w_\tau^j, z_\tau^j)\}_{j=1}^{J_\tau}$, with $u_\tau^j \in W^{1,\gamma}(\Omega\backslash\Gamma_C;\mathbb{R}^d)$, $w_\tau^j \in W^{1,2}(\Omega\backslash\Gamma_C)$, and $z_\tau^j \in H^1(\Gamma_C)$, fulfilling for all $j = 1, ..., J_\tau$ the recursive scheme consisting of

- the (boundary-value problem for the) **discrete momentum equation**:

$$\begin{aligned}\int_{\Omega\backslash\Gamma_C} \big(\mathrm{DR}_2(e(\mathrm{D}_t u_\tau^j)) + \mathrm{DW}_2(e(u_\tau^j)) - \mathbb{B}\Theta(w_\tau^j) + \mathrm{DW}_p(e(u_\tau^j)) + \tau\mathrm{DW}_\gamma(e(u_\tau^j))\big) : e(v - u_\tau^j)\,\mathrm{d}x \\ + \int_{\Gamma_C} k z_\tau^j [\![u_\tau^j]\!]\cdot[\![v - u_\tau^j]\!]\,\mathrm{d}S \geq \langle \mathrm{F}_\tau^j, v - u_\tau^j\rangle \\ \text{for all } v \in W^{1,p}(\Omega\backslash\Gamma_C;\mathbb{R}^d) \text{ with } [\![v(x)]\!] \in \mathrm{C}(x) \text{ for a.a. } x \in \Gamma_C,\end{aligned}\tag{A.4}$$

where we use the notation $\mathrm{D}_t u_\tau^j := \frac{u_\tau^j - u_\tau^{j-1}}{\tau}$ and $\mathrm{W}_\gamma(e) := \frac{1}{\gamma}|e|^\gamma$;

- the (boundary-value problem for the) **discrete enthalpy equation**: for all $\zeta \in W^{1,2}(\Omega\backslash\Gamma_C)$

$$\begin{aligned}\int_{\Omega\backslash\Gamma_C} \mathrm{D}_t w_\tau^j \zeta\,\mathrm{d}x + \int_{\Omega\backslash\Gamma_C} \mathcal{K}(e(u_\tau^j), w_\tau^j)\nabla w_\tau^j\cdot\nabla\zeta\,\mathrm{d}x + \int_{\Sigma_C} \eta([\![u_\tau^{j-1}]\!], z_\tau^j)[\![\Theta(w_\tau^j)]\!]\,[\![\zeta]\!]\,\mathrm{d}S \\ = \int_{\Omega\backslash\Gamma_C} \big(2\mathrm{R}_2(e(\mathrm{D}_t u_\tau^j)) - \Theta(w_\tau^j)\mathbb{B}:e(\mathrm{D}_t u_\tau^j)\big)\,\zeta\,\mathrm{d}x - \int_{\Gamma_C} \frac{\zeta|_{\Gamma_C}^+ + \zeta|_{\Gamma_C}^-}{2}\, a_1 \mathrm{D}_t z_\tau^j\,\mathrm{d}S + \langle \mathrm{H}_\tau^j, \zeta\rangle;\end{aligned}\tag{A.5}$$

- the **time-incremental minimization** problem for the delamination parameter

$$z_\tau^j \in \mathrm{Argmin}_{z\in H^1(\Gamma_C)} \left\{\tau\mathcal{R}_1\left(\frac{z - z_\tau^{j-1}}{\tau}\right) + \Phi_{k,m}(u_\tau^{j-1}, z)\right\}. \tag{A.6}$$

Remark A.2 We highlight that time-incremental minimization (A.6) is decoupled from equations (A.4)–(A.5): indeed, starting from $(u_\tau^{j-1}, z_\tau^{j-1})$ one first solves (A.6) and then plugs z_τ^j in system (A.4)–(A.5), which can be handled via the theory of pseudo-monotone operators. The carefully designed coupling between (A.4)–(A.5) and (A.6) will be heavily exploited in the proof of Lemma A.14 below. Observe that the Euler-Lagrange equation for (A.6) is indeed the *discrete* version of the flow rule (2.14), viz.

$$\partial\mathscr{F}(z_\tau^{j-1}; z_\tau^j) + \frac{1}{2}k\big|[\![u_\tau^j]\!]\big|^2 + mg'(z_\tau^j) - \frac{1}{m}\Delta z_\tau^j - a_0 - a_1 \ni 0 \quad \text{a.e. in } \Gamma_C, \tag{A.7}$$

with $\mathscr{F}(z_\tau^{j-1}; z) = \int_{\Gamma_C} \Big(I_{(-\infty,0]}\big(\frac{z - z_\tau^{j-1}}{\tau}\big) + I_{[0,1]}(z)\Big)\,\mathrm{d}S$ and $\partial\mathscr{F}(z_\tau^{j-1};\cdot) : L^2(\Gamma_C) \rightrightarrows L^2(\Gamma_C)$ its subdifferential. However, (A.6) and (A.7) are not equivalent because of the nonconvexity of g, which brings about additional analytical difficulties with respect to the adhesive contact systems considered in [RR11b, RR11a].

Lemma A.3 *Assume* (3.7), (3.8), (3.11), (3.13). *Then, Problem A.1 admits at least one solution.*

Sketch of the proof. The existence of a solution z_τ^j to (A.6) follows from the lower semicontinuity and coercivity properties of the functional $\Phi_{k,m}$, via the *direct method* in the Calculus of Variations. We then plug z_τ^j in (A.4)–(A.5) and prove the existence of solutions by suitably adapting the argument for [RR11b, Lemma 7.4], where the time-discretization scheme for a thermal adhesive contact model similar to the Modica-Mortola system was analyzed.

The key idea is to apply to the *elliptic* system (A.4)–(A.5) a Leray-Lions type existence theorem (see, e.g., [Rou05, Chap. 2]). To do so, one needs to verify that the main part of the (pseudo-monotone) operator involved in (A.4)–(A.5), is strictly monotone, and that said operator is coercive in the space $W^{1,\gamma}(\Omega\backslash\Gamma_C;\mathbb{R}^d)\times W^{1,2}(\Omega\backslash\Gamma_C)$ for the unknown (u,w). For this coercivity property, the term $\tau W_\gamma(e(u)) = \tau|e(u)|^{\gamma-2}e(u)$ in the discrete momentum equation plays a crucial role, as it compensates the growth of the quadratic terms on the left-hand side of (A.5) with the right-hand side of (A.4). Indeed, in order to prove the coercivity of the operator underlying (A.4)–(A.5), it is necessary to test (A.5) by w_τ^j, and from this derive a bound for $\|w_\tau^j\|_{W^{1,2}(\Omega\backslash\Gamma_C)}$. The respective calculations involve the following estimate

$$\begin{aligned}\left|\int_{\Omega\backslash\Gamma_C}\Theta(w_\tau^j)\mathbb{B}{:}e(\mathrm{D}_t u_\tau^j)w_\tau^j\,\mathrm{d}x\right| &\le \frac{1}{16\tau}\|w_\tau^j\|_{L^2(\Omega)}^2 + C\int_{\Omega\backslash\Gamma_C}\left|e(u_\tau^j)-e(u_\tau^{j-1})\right|^2\left|(w_\tau^j)^{2/\omega}+1\right|\mathrm{d}x\\ &\le \frac{1}{8\tau}\|w_\tau^j\|_{L^2(\Omega)}^2 + C\left(\|e(u_\tau^j)\|_{L^{p_\omega}(\Omega;\mathbb{R}^{d\times d})}^{p_\omega} + \|e(u_\tau^{j-1})\|_{L^{p_\omega}(\Omega;\mathbb{R}^{d\times d})}^{p_\omega}+1\right)\\ &\le \frac{1}{8\tau}\|w_\tau^j\|_{L^2(\Omega)}^2 + \frac{\tau}{8C}\|u_\tau^j\|_{W^{1,\gamma}(\Omega\backslash\Gamma_C;\mathbb{R}^d)}^\gamma + C\left(\|u_\tau^{j-1}\|_{W^{1,\gamma}(\Omega\backslash\Gamma_C;\mathbb{R}^d)}^\gamma+1\right),\end{aligned} \tag{A.8}$$

where we have used the placeholder $p_\omega := \frac{2\omega}{\omega-1}$. In (A.8), the first estimate is due to Hölder's inequality and to the growth condition (3.9) for Θ, the second one again derives from Hölder's and Young's inequalities, and the last one also from the fact that $\gamma > p_\omega$. Therefore, we can absorb the second term on the right-hand side of (A.8) into the left-hand side of the discrete momentum equation tested by u_τ^j, whereas the first summand is estimated by the left-hand side of (A.5) tested by w_τ^j. The term involving $\|u_\tau^{j-1}\|_{W^{1,\gamma}(\Omega;\mathbb{R}^d)}^\gamma$ is estimated from the previous step. With analogous calculations one deals with the term $\left|\int_\Omega 2\mathrm{R}_2(e(\mathrm{D}_t u_\tau^j))w_\tau^j\,\mathrm{d}x\right|$. The reader is referred to the proof of [RR11b, Lemma 7.4] for all details. ■

We now introduce the interpolants of the discrete solutions $\{(u_\tau^j, w_\tau^j, z_\tau^j)\}_{j=1}^{J_\tau}$.

Notation A.4 (Interpolants) For $\tau > 0$ fixed, the left-continuous and right-continuous *piecewise constant*, and the *piecewise linear* interpolants of the family $\{u_\tau^j\}_{j=1}^{J_\tau}$ are respectively the functions $\overline{u}_\tau$, $\underline{u}_\tau$, $u_\tau : (0,T) \to W_{\Gamma_D}^{1,\gamma}(\Omega\backslash\Gamma_C;\mathbb{R}^d)$ defined by

$$\overline{u}_\tau(t) = u_\tau^j, \qquad \underline{u}_\tau(t) = u_\tau^{j-1}, \qquad u_\tau(t) = \frac{t-t_\tau^{j-1}}{\tau}u_\tau^j + \frac{t_\tau^j - t}{\tau}u_\tau^{j-1} \qquad \text{for } t \in (t_\tau^{j-1}, t_\tau^j]. \tag{A.9}$$

In the same way, we denote by $\overline{w}_\tau$, $\underline{w}_\tau$, $\overline{z}_\tau$ and $\underline{z}_\tau$, the piecewise constant interpolants of the elements $\{w_\tau^j\}_{j=1}^{J_\tau}$ and $\{z_\tau^j\}_{j=1}^{J_\tau}$, and by w_τ and z_τ the related piecewise linear interpolants. We shall also consider the interpolants $\overline{\mathrm{F}}_\tau$ and $\overline{\mathrm{H}}_\tau$ of the J_τ-tuples $\{\mathrm{F}_\tau^j\}_{j=1}^{J_\tau}$ and $\{\mathrm{H}_\tau^j\}_{j=1}^{J_\tau}$. Finally, we use the notation $\bar{\mathfrak{t}}_\tau$ for the left-continuous piecewise constant interpolant associated with the partition, i.e. $\bar{\mathfrak{t}}_\tau(t) = t_\tau^j$ if $t_\tau^{j-1} < t \le t_\tau^j$.

Lemma A.5 *Assume* (3.7), (3.8), (3.11), (3.13). *Define* $\Phi_\tau(u,z) := \Phi_{k,m}(u,z) + \tau\int_{\Omega\backslash\Gamma_C} \mathrm{W}_\gamma(e(u))\,\mathrm{d}x$. *Then, for all* $\tau > 0$ *the approximate solutions* $(\overline{u}_\tau, \underline{u}_\tau, \overline{w}_\tau, \overline{z}_\tau, u_\tau, w_\tau, z_\tau)$ *fulfill the "discrete mechanical energy" inequality*

$$\begin{aligned}&\Phi_\tau\big(\overline{u}_\tau(t), \overline{z}_\tau(t)\big) + \int_0^{\bar{\mathfrak{t}}_\tau(t)}\left(\int_{\Omega\backslash\Gamma_C} 2\mathrm{R}_2\big(e(\dot{u}_\tau)\big) + \int_{\Gamma_C} a_1|\dot{z}_\tau|\,\mathrm{d}S\right)\mathrm{d}s\\ &\le \Phi_\tau\big(u_{0,\tau}, z_0\big) + \int_0^{\bar{\mathfrak{t}}_\tau(t)}\left(\int_{\Omega\backslash\Gamma_C}\Theta(\overline{w}_\tau)\mathbb{B}{:}e\big(\dot{u}_\tau\big)\,\mathrm{d}x + \langle\overline{\mathrm{F}}_\tau, \dot{u}_\tau\rangle\right)\mathrm{d}s,\end{aligned} \tag{A.10}$$

and the "discrete semistability" for a.a. $t \in (0,T)$

$$\Phi_\tau\big(\underline{u}_\tau(t), \overline{z}_\tau(t)\big) \le \Phi_\tau\big(\underline{u}_\tau(t), \tilde{z}\big) + \mathcal{R}_1(\tilde{z} - \overline{z}_\tau(t)) \quad \textit{for all } \tilde{z} \in \mathcal{Z}_{\mathrm{MM}} \textit{ with } \tilde{z} \le \underline{z}_\tau(t) \textit{ on } \Gamma_C. \tag{A.11}$$

Proof: For notational simplicity we will develop the calculations in terms of the *discrete* solutions $\{(u_\tau^j, w_\tau^j, z_\tau^j)\}_{j=1}^{J_\tau}$. It follows from the time-incremental minimization (A.6) and the definition (3.16) of $\Phi_{k,m}$ that $z_\tau^j \le z_\tau^{j-1}$ a.e. on $\Gamma_{\rm C}$, and

$$\mathcal{R}_1(z_\tau^j - z_\tau^{j-1}) + \int_{\Gamma_{\rm C}} \big(\tfrac{k}{2} z_\tau^j |[\![u_\tau^{j-1}]\!]|^2 - a_0 z_\tau^j\big)\,{\rm d}S + \mathcal{G}_m(z_\tau^j) \le \int_{\Gamma_{\rm C}} \big(\tfrac{k}{2} z_\tau^{j-1} |[\![u_\tau^{j-1}]\!]|^2 - a_0 z_\tau^{j-1}\big)\,{\rm d}S + \mathcal{G}_m(z_\tau^{j-1}). \tag{A.12}$$

Now, let us choose in (A.4) the (admissible) test function $v = u_\tau^{j-1}$ and change sign in the inequality. Then, we use the elementary estimates ${\rm DR}_2(e({\rm D}_t u_\tau^j)){:}e(u_\tau^j - u_\tau^{j-1}) = \tau 2{\rm R}_2(e({\rm D}_t u_\tau^j))$ as well as ${\rm DW}_n(e(u_\tau^j)){:}e(u_\tau^j - u_\tau^{j-1}) \ge {\rm W}_n(e(u_\tau^j)) - {\rm W}_n(e(u_\tau^{j-1}))$ for $n = 2, p, \gamma$, and $k z_\tau^j [\![u_\tau^j]\!] \cdot [\![u_\tau^j - u_\tau^{j-1}]\!] \ge \frac{k}{2} z_\tau^j |[\![u_\tau^j]\!]|^2 - \frac{k}{2} z_\tau^j |[\![u_\tau^{j-1}]\!]|^2$. Thus, we obtain

$$\begin{aligned}
&\int_{\Omega\setminus\Gamma_{\rm C}} \big({\rm W}_2(e(u_\tau^j)) + {\rm W}_p(e(u_\tau^j)) + \tau {\rm W}_\gamma(e(u_\tau^j))\big)\,{\rm d}x + \tau \int_{\Omega\setminus\Gamma_{\rm C}} 2{\rm R}_2(e({\rm D}_t u_\tau^j))\,{\rm d}x + \int_{\Gamma_{\rm C}} \tfrac{k}{2} z_\tau^j |[\![u_\tau^j]\!]|^2 \\
&\int_{\Omega\setminus\Gamma_{\rm C}} \big({\rm W}_2(e(u_\tau^{j-1})) + {\rm W}_p(e(u_\tau^{j-1})) + \tau {\rm W}_\gamma(e(u_\tau^{j-1}))\big)\,{\rm d}x + \tau \int_{\Omega\setminus\Gamma_{\rm C}} \Theta(w_\tau^j)\mathbb{B}{:}e({\rm D}_t u_\tau^j)\,{\rm d}x \\
&\qquad + \int_{\Gamma_{\rm C}} \tfrac{k}{2} z_\tau^j |[\![u_\tau^{j-1}]\!]|^2 + \tau \langle {\rm F}_\tau^j, {\rm D}_t u_\tau^j \rangle\,.
\end{aligned} \tag{A.13}$$

Hence, we add (A.12) and (A.13), observing that the term $\int_{\Gamma_{\rm C}} \frac{k}{2} z_\tau^j |[\![u_\tau^{j-1}]\!]|^2\,{\rm d}S$ cancels out. Upon summing over the index j, we thus arrive at the discrete mechanical energy inequality (A.10).

From (A.6) it also follows that

$$\mathcal{R}_1(z_\tau^j - z_\tau^{j-1}) + \Phi_{k,m}(u_\tau^{j-1}, z_\tau^j) \le \mathcal{R}_1(\tilde z - z_\tau^{j-1}) + \Phi_{k,m}(u_\tau^{j-1}, \tilde z)$$

for all $\tilde z \in H^1(\Gamma_{\rm C})$ with $\tilde z \le z_\tau^{j-1}$ on $\Gamma_{\rm C}$, whence we immediately conclude (A.11). ■

As a consequence of Lemma A.5, we have the following result.

Proposition A.6 (A priori estimates) *Assume* (3.7), (3.8), (3.11), *and let* (u_0, z_0, θ_0) *be a triple of initial data complying with* (3.13) *and the semistability condition* (4.3). *Then, there exist constants* $S^0 > 0$ *and, for every* $1 \le r < \frac{d+2}{d+1}$, $S_r^0 > 0$, *such that for all* τ, m, $k > 0$ *and for all approximate solutions* $(\overline{u}_\tau, \overline{w}_\tau, \overline{z}_\tau, u_\tau, w_\tau, z_\tau)$ *the following estimates hold*

$$\|\overline{u}_\tau\|_{L^\infty(0,T;W^{1,p}_{\Gamma_{\rm D}}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))} + \|u_\tau\|_{L^\infty(0,T;W^{1,p}_{\Gamma_{\rm D}}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))} + \|u_\tau\|_{W^{1,2}(0,T;W^{1,2}_{\Gamma_{\rm D}}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))} \le S_0\,, \tag{A.14a}$$

$$\|\overline{u}_\tau\|_{L^\infty(0,T;W^{1,\gamma}_{\Gamma_{\rm D}}(\Omega\setminus\Gamma_{\rm C};\mathbb{R}^d))} \le \frac{S_0}{\sqrt[\gamma]{\tau}}, \tag{A.14b}$$

$$\sup_{t\in[0,T]} \Phi_\tau\left(\overline{u}_\tau(t), \overline{z}_\tau(t)\right) \le S_0, \tag{A.14c}$$

$$\|\overline{z}_\tau\|_{L^\infty(\Sigma_{\rm C})} + \|z_\tau\|_{{\rm BV}([0,T];L^1(\Gamma_{\rm C}))} \le S_0\,, \tag{A.14d}$$

$$\|\overline{w}_\tau\|_{L^\infty(0,T;L^1(\Omega))} + \|\dot w_\tau\|_{L^1(0,T;W^{1,r'}(\Omega\setminus\Gamma_{\rm C})^*)} \le S_0, \tag{A.14e}$$

$$\|w_\tau\|_{L^r(0,T;W^{1,r}(\Omega\setminus\Gamma_{\rm C}))} \le S_r \quad \textit{for any } 1 \le r < \tfrac{d+2}{d+1} \tag{A.14f}$$

where $r' = \frac{r}{r-1}$ *is the conjugate exponent of* r. *Estimates* (A.14d), (A.14e), (A.14f) *respectively hold for* z_τ, $\overline{z}_\tau$, w_τ *and* $\overline{w}_\tau$, *as well.*

The *proof* relies on the energy inequality (A.10) and on a suitable test of the discrete enthalpy equation (A.5). The calculations are identical to those performed for [RR11b, Lemma 7.7], to which the reader is referred. We can now develop the

Proof of Theorem 4.2. We follow the steps outlined in Sec. 3.4. However, we only detail the passage to the limit in the discrete semistability condition (A.11), since the remaining steps can be performed as in the proof of [RR11b, Thm. 6.1], see also the arguments developed here in Section 4.

Step 0: selection of converging subsequences. Let $(\tau_j)_j$ be a vanishing sequence of time-steps. Arguing in the very same way as in the proof of Thm. 4.3, it can be checked that, there exists a triple (u,w,z) such that, up to a (not relabeled) subsequence, for the approximate solutions of Problem A.1 (cf. Notation A.9), the following convergences hold as $j\to\infty$:

$$\begin{aligned} &u_{\tau_j}\rightharpoonup u \quad \text{in } L^\infty(0,T;W^{1,p}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d))\cap W^{1,2}(0,T;W^{1,2}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)),\\ &u_{\tau_j}\to u \quad \text{in } \mathrm{C}^0([0,T];W^{1-\epsilon,p}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)) \end{aligned} \tag{A.15}$$

$$\begin{aligned} &\overline{u}_{\tau_j}\overset{*}{\rightharpoonup} u \ \text{ in } L^\infty(0,T;W^{1,p}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)),\quad \overline{u}_{\tau_j}\to u \ \text{ in } L^\infty(0,T;W^{1-\epsilon,p}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)),\\ &\overline{u}_{\tau_j}(t)\to u(t) \ \text{ in } W^{1-\epsilon,p}_{\Gamma_\mathrm{D}}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)) \text{ for all } t\in[0,T] \end{aligned} \tag{A.16}$$

and for all $\epsilon\in(0,1]$. Besides, (A.14b) yields that

$$\tau_j\big\||e(\overline{u}_{\tau_j})|^{\gamma-2}e(\overline{u}_{\tau_j})\big\|_{L^{\gamma/(\gamma-1)}((0,T)\times\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^{d\times d})}\le S_0\tau_j^{1/\gamma}\to 0 \quad \text{as } \tau_j\to 0. \tag{A.17}$$

Furthermore, taking into account estimate (A.14c) and the fact that $z\mapsto\Phi_\tau(u,z)$ has bounded sublevels in $H^1(\Gamma_\mathrm{C})$, and using an infinite-dimensional version of Helly's principle (see e.g. [MT04, Thm. 6.1]), we find that there exists $z\in L^\infty(0,T;H^1(\Gamma_\mathrm{C}))\cap \mathrm{BV}([0,T];L^1(\Gamma_\mathrm{C}))$, with $0\le z(t,x)\le 1$ for almost all $(t,x)\in\Sigma_\mathrm{C}$, such that

$$\overline{z}_{\tau_j},\,\underline{z}_{\tau_j}\overset{*}{\rightharpoonup} z \ \text{ in } L^\infty(0,T;H^1(\Gamma_\mathrm{C})),\qquad \overline{z}_{\tau_j}(t),\,\underline{z}_{\tau_j}(t)\overset{*}{\rightharpoonup} z(t) \ \text{ in } H^1(\Gamma_\mathrm{C}) \text{ for all } t\in[0,T]. \tag{A.18}$$

On account of the compact embedding $H^1(\Gamma_\mathrm{C})\Subset L^q(\Gamma_\mathrm{C})$ for all $1\le q<\infty$, we also have

$$\overline{z}_{\tau_j}(t),\ \underline{z}_{\tau_j}(t)\to z(t) \ \text{ in } L^q(\Gamma_\mathrm{C}) \text{ for all } t\in[0,T] \text{ and } 1\le q<\infty, \text{ whence} \tag{A.19}$$

$$\mathrm{Var}_{\mathcal{R}_1}(z;[s,t])=\lim_{\tau_j\to0}\int_s^t\int_{\Gamma_\mathrm{C}}a_1|\dot{z}_{\tau_j}(r)|\,\mathrm{d}S\mathrm{d}r \quad \text{for all } 0\le s\le t\le T \tag{A.20}$$

(recall definition (3.31) of $\mathrm{Var}_{\mathcal{R}_1}$). Thirdly, by the same tokens we conclude that there exists $w\in L^r(0,T;W^{1,r}(\Omega\backslash\Gamma_\mathrm{C}))\cap\mathrm{BV}([0,T];W^{1,r'}(\Omega\backslash\Gamma_\mathrm{C})^*)$ such that

$$\begin{aligned} &\overline{w}_{\tau_j},\,w_{\tau_j}\rightharpoonup w \quad \text{in } L^r(0,T;W^{1,r}(\Omega\backslash\Gamma_\mathrm{C})),\\ &\overline{w}_{\tau_j},\,w_{\tau_j}\to w \quad \text{in } L^r(0,T;W^{1-\epsilon,r}(\Omega\backslash\Gamma_\mathrm{C}))\cap L^q(0,T;L^1(\Omega)) \ \ \forall\,\epsilon\in(0,r-1],\ 1\le q<\infty, \end{aligned} \tag{A.21}$$

$$w_{\tau_j}(t)\overset{*}{\rightharpoonup} w(t) \quad \text{in } W^{1,r'}(\Omega\backslash\Gamma_\mathrm{C})^* \text{ for all } t\in[0,T]. \tag{A.22}$$

Finally, let us observe that, thanks to (A.18) and (A.19), we have $\mathcal{G}_m(z(t))\le\liminf_{\tau_j\to0}\mathcal{G}_m(\overline{z}_{\tau_j}(t))$ for all $t\in[0,T]$. Therefore, also in view of the previous convergences (A.15)–(A.17), we conclude

$$\Phi_{k,m}(u(t),z(t))\le\liminf_{\tau_j\to0}\Phi_{\tau_j}(\overline{u}_{\tau_j}(t),\overline{z}_{\tau_j}(t)). \tag{A.23}$$

Step 1: momentum equation. Relying on convergences (A.15)–(A.17), (A.18), (A.19)–(A.21), as well as on the convergence $\overline{\mathrm{F}}_\tau\to\mathrm{F}$ in $L^2(0,T;W^{1,2}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)^*)\cap W^{1,1}(0,T;W^{1,p}(\Omega\backslash\Gamma_\mathrm{C};\mathbb{R}^d)^*)$, and arguing in the very same way as in the proof of Step 1 for Theorem 4.3, it is possible to pass to the limit in the discrete momentum inclusion (A.4) for the approximating solutions, and conclude that (u,w,z) comply with the weak formulation (3.28a) of the momentum inclusion in the adhesive case.

Step 2: semistability condition. Like in the proof of Thm. 4.3 and Thm. 5.1, in order to prove that the pair (u,z) fulfills the semistability condition (3.29), we need to verify for the sequence $(\underline{u}_{\tau_j},\overline{z}_{\tau_j})_j$ the *mutual recovery sequence* condition. Viz., that for all $t\in[0,T]$ and for all $\tilde z\in\mathcal{Z}_{\mathrm{MM}}=H^1(\Gamma_\mathrm{C})$ with $\mathcal{R}_1(\tilde z-z)<\infty$, there is a sequence $(\tilde z_j)_j$ (t-dependence omitted) so that $\tilde z_j\rightharpoonup\tilde z$ in $H^1(\Gamma_\mathrm{C})$ as $j\to\infty$ and

$$\limsup_{\tau_j\to0}\big(\Phi_{\tau_j}(\underline{u}_{\tau_j}(t),\tilde z_j)+\mathcal{R}_1(\tilde z_j-\overline{z}_{\tau_j}(t))-\Phi_{\tau_j}(\underline{u}_{\tau_j}(t),\overline{z}_{\tau_j}(t))\big)\le\Phi_{k,m}(u(t),\tilde z)+\mathcal{R}_1(\tilde z-z(t))-\Phi_{k,m}(u(t),z(t)). \tag{A.24}$$

Observe that the left-hand side of (A.24) is positive thanks to (A.11). Notice that, for (A.24) to hold, it is necessary that $\tilde z_j\in H^1(\Gamma_\mathrm{C})\cap L^\infty(\Gamma_\mathrm{C})$ and

$$0\le\tilde z_j\le\underline{z}_{\tau_j}(t)\le 1 \quad \text{a.e. in } \Gamma_\mathrm{C}. \tag{A.25}$$

For $(\tilde{z}_j)_j$, we use the construction from the proof of [TM10, Thm. 3.14], and set

$$\tilde{z}_j := \min\{(\tilde{z}-\delta_j)^+, \underline{z}_{\tau_j}(t)\} = \begin{cases} \tilde{z}-\delta_j & \text{if } (\tilde{z}-\delta_j)^+ \leq \underline{z}_{\tau_j}(t), \\ \underline{z}_{\tau_j}(t) & \text{if } (\tilde{z}-\delta_j)^+ > \underline{z}_{\tau_j}(t) \end{cases} \quad \text{with } \delta_j := \|\underline{z}_{\tau_j}(t)-z(t)\|_{L^2(\Gamma_C)}^{1/2}. \tag{A.26}$$

Clearly, $(\tilde{z}_j)_j$ fulfill (A.25). In view of (A.19), $\delta_j \to 0$ as $j \to \infty$. Let us now verify that $(\tilde{z}_j)_j$ complies with (A.24). First of all, the very same argument as in [TM10] yields that $(\tilde{z}_j)_j \subset H^1(\Gamma_C)$, and that $\tilde{z}_j \rightharpoonup \tilde{z}$ in $H^1(\Gamma_C)$, hence $\tilde{z}_j \to \tilde{z}$ in $L^q(\Gamma_C)$ for all $1 \leq q < \infty$. Therefore, on account of (A.18) we immediately have that $\lim_{\tau_j \to 0} \mathcal{R}_1(\tilde{z}_j - \overline{z}_{\tau_j}(t))) = \mathcal{R}_1(\tilde{z}-z(t))$. Furthermore, also in view of (A.16) we have

$$\begin{cases} \lim_{\tau_j\to 0} \int_{\Gamma_C} \frac{k}{2}\left(\tilde{z}_j-\overline{z}_{\tau_j}(t)\right) |[\![\underline{u}_{\tau_j}(t)]\!]|^2 \,\mathrm{d}S = \int_{\Gamma_C} \frac{k}{2}\left(\tilde{z}-z(t)\right) |[\![u(t)]\!]|^2 \,\mathrm{d}S, \\ \lim_{\tau_j\to 0} \int_{\Gamma_C} a_0(\overline{z}_{\tau_j}(t)-\tilde{z}_j)\,\mathrm{d}S = \int_{\Gamma_C} a_0(z(t)-\tilde{z})\,\mathrm{d}S, \\ \lim_{\tau_j\to 0} \int_{\Gamma_C} m\left(g(\tilde{z}_j)-g(\overline{z}_{\tau_j}(t))\right)\,\mathrm{d}S = \int_{\Gamma_C} m\left(g(\tilde{z})-g(z(t))\right)\,\mathrm{d}S \end{cases} \tag{A.27}$$

with $g(z) = z^2(1-z)^2$. Repeating the very same calculations as for [TM10, Thm. 3.14], it can also be checked that

$$\limsup_{\tau_j\to 0} \int_{\Gamma_C} \frac{1}{2m}\left(|\nabla\tilde{z}_j|^2-|\nabla\overline{z}_{\tau_j}(t)|^2\right)\,\mathrm{d}S \leq \int_{\Gamma_C} \frac{1}{2m}\left(|\nabla\tilde{z}|^2-|\nabla z(t)|^2\right)\,\mathrm{d}S. \tag{A.28}$$

Then, (A.24) ensues from (A.27) and (A.28).
Step 3: mechanical energy inequality. The mechanical energy inequality (3.30) can be obtained via the very same lower semicontinuity argument as in Step 3 of the proof of Thm. 4.3.
Steps 4: enthalpy inequality. The previously proved convergences, as well as the fact that $\overline{\mathrm{H}}_{\tau_j} \to \mathrm{H}$ in $L^1(0,T; W^{1,r}(\Omega\backslash\Gamma_C\ \mathbb{R}^d)^*)$, allow us to take the limit of the discrete enthalpy equation (A.5). Arguing in the very same way as in Step 4 of the proof of Thm. 4.3, we prove the weak enthalpy inequality (3.32).
Positivity of the temperature. Repeating the arguments of the proofs of [RR11b, Lemma 7.4, Thm. 5.1], it is possible to show that, if there exists $\theta^* > 0$ such that $\theta_0(x) \geq \theta^*$ for almost all $x \in \Omega$, then

$$w(x,t) \geq \frac{1}{C'T + h(\theta^*) + 1} \quad \text{for a.a. } (x,t) \in \Omega \times (0,T), \tag{A.29}$$

where the constant C' only depends on the problem data. Then, (4.2) ensues. ∎

A.2 Tools from the theory of BV-functions

In order to make this paper as self-contained as possible, below we collect all the measure-theoretic definitions and tools from the theory of BV-functions which have been used. In what follows, $D \subset \mathbb{R}^m$ will denote a bounded set and $\mathcal{X}_D$, with $\mathcal{X}_D(x) = 1$ if $x \in D$, $\mathcal{X}_D(x) = 0$ otherwise, its characteristic function. In Sections 1–6, all the statements below apply to $D = \Gamma_C$ and $m = d-1$.

Definition A.7 ([AFP05, Def. 3.35] Sets of finite perimeter) *Let E be an $\mathcal{L}^m$-measurable subset of $\mathbb{R}^m$. For any open set $D \subset \mathbb{R}^m$ the perimeter of E in D, denoted by $P(E,D)$, is the variation of the characteristic function $\mathcal{X}_E$ in D, i.e.*

$$P(E,D) := \sup\left\{ \int_E \operatorname{div}\varphi\,\mathrm{d}x \,\middle|\, \varphi \in \mathrm{C}^1_c(D)^m,\ \|\varphi\|_{L^\infty(D)} \leq 1 \right\}. \tag{A.30}$$

We say that E is a set of finite perimeter in D if $P(E,D) < \infty$. Here, $\mathrm{C}^1_c(D)^m$ is the space of continuously differentiable functions $v : D \to \mathbb{R}^m$ with compact support in D.

Theorem A.1 ([AFP05, Th. 3.36]) *For any set E of finite perimeter in D the distributional derivative $\mathrm{D}\mathcal{X}_E$ is an $\mathbb{R}^m$-valued finite Radon measure in D. Moreover, $P(E,D) = |\mathrm{D}\mathcal{X}_E|(D)$ and a generalized Gauss-Green formula holds:*

$$\int_E \operatorname{div}\varphi\,\mathrm{d}x = -\int_D \nu_E\cdot\varphi\,\mathrm{d}|\mathrm{D}\mathcal{X}_E| \quad \textit{for all } \mathrm{C}^1_c(D)^m, \tag{A.31}$$

where $\mathrm{D}\mathcal{X}_E = \nu_E|\mathrm{D}\mathcal{X}_E|$ is the polar decomposition of $\mathrm{D}\mathcal{X}_E$, i.e. $\nu_E \in L^1(D,|\mathrm{D}\mathcal{X}_E|)^m$ is the Radon-Nikodým density for the measure $\mathrm{D}\mathcal{X}_E$ with respect to the measure $|\mathrm{D}\mathcal{X}_E|$.

Proposition A.8 ([AFP05, Prop. 3.38] Properties of the perimeter)

1. *The mapping* $E \mapsto P(E,D)$ *is lower semicontinuous w.r.t. local convergence in measure in* D.
2. *The mapping* $E \mapsto P(E,D)$ *is local, i.e.* $P(E,D) = P(F,D)$ *whenever* $|D \cap \big((E \backslash F) \cup (F \backslash E)\big| = 0$.
3. *It holds* $P(E,D) = P(\mathbb{R}^m \backslash E, D)$ *and*

$$P(E \cup F, D) + P(E \cap F, D) \leq P(E,D) + P(F,D)\,. \tag{A.32}$$

Theorem A.2 ([AFP05, Th. 3.40] Coarea formula in BV**)** *For* $v \in L^1_{\mathrm{loc}}(D)$ *the varation on any open set* $D \subset \mathbb{R}^m$ *is defined by* $V(v,D) := \sup\big\{ \int_D u \operatorname{div} \varphi \, \mathrm{d}x \,\big|\, \varphi \in \mathrm{C}^1_c(D)^m,\ \|\varphi\|_\infty = 1 \big\}$. *It holds*

$$V(v,D) = \int_{-\infty}^{\infty} P(\{x \in D \,|\, v(x) > t\}, D)\,\mathrm{d}t\,. \tag{A.33}$$

In particular, if $v \in BV(D)$ *the set* $\{v > t\}$ *has finite perimeter for* $\mathcal{L}^1$*-a.a.* $t \in \mathbb{R}$ *and*

$$|\mathrm{D}v|(B) = \int_{-\infty}^{\infty} |\mathrm{D}\mathcal{X}_{\{v>t\}}|(B)\,\mathrm{d}t\,, \quad \mathrm{D}v(B) = \int_{-\infty}^{\infty} \mathrm{D}\mathcal{X}_{\{v>t\}}(B)\,\mathrm{d}t \tag{A.34}$$

for any Borel set $B \subset D$.

Definition A.9 ([AFP05, Def. 3.54] Reduced boundary) *Let* E *be an* $\mathcal{L}^m$*-measurable subset of* $\mathbb{R}^m$ *and* D *the largest open set such that* E *is locally of finite perimeter in* D. *We define the reduced boundary* $\mathfrak{F}E$ *as the collection of all points* $x \in \operatorname{supp} |\mathrm{D}\mathcal{X}_E| \cap D$ *such that the limit*

$$\nu_E(x) := \lim_{\varrho \to 0} \frac{\mathrm{D}\mathcal{X}_E(B_\varrho(x))}{|\mathrm{D}\mathcal{X}_E|(B_\varrho(x))} \tag{A.35}$$

exists in $\mathbb{R}^m$ *and satisfies* $\nu_E(x) = 1$. *The function* $\nu_E : \mathfrak{F}E \to \mathbb{S}^{m-1}$ *is called the generalized inner normal to* E; *here,* $\mathbb{S}^{m-1}$ *denotes the unit sphere in* $\mathbb{R}^m$.

Definition A.10 ([AFP05, Def. 3.60] Points of density t**, essential boundary)** *For all* $t \in [0,1]$ *and every* $\mathcal{L}^m$*-measurable set* $E \subset \mathbb{R}^m$ *we denote by* E^t *the set*

$$\left\{ x \in \mathbb{R}^m \,\Big|\, \lim_{\varrho \to 0} \frac{\mathcal{L}^m(E \cap B_\varrho(x))}{\mathcal{L}^m(B_\varrho(x))} = t \right\} \tag{A.36}$$

of all points where E *has density* t. *We denote by* $\partial^* E$ *the essential boundary of* E, *i.e. the set* $\mathbb{R}^m \backslash (E^0 \cup E^1)$ *of points where the density is either* 0 *or* 1. *Moreover,* E^1 *can be considered as the measure-theoretic interior and* E^0 *as the measure-theoretic exterior of the set* E.

Corollary A.11 *The measure-theoretic interior has the following properties:*

1. *Let* $N \subset D$ *with* $\mathcal{L}^m(N) = 0$. *Then* $N^1 = \emptyset$ *and* $(D \backslash N)^1 = D^1$.
2. *Let* $A \subset B \subset D$. *Then* $A^1 \subset B^1 \subset D^1$.

Proof: Ad 1.): Assume that $x \in N^1$. Since $\mathcal{L}^m(N) = 0$ it is $\mathcal{L}^m(N \cap B_\varrho(x)) = 0$ for all $\varrho > 0$. Hence $x \in N^0$ in contradiction to the assumption and thus $N^1 = \emptyset$.

Let now $x \in D^1$ and assume that $x \notin (D \backslash N)^1$, i.e. there is an index $\varrho_0 > 0$ such that $\mathcal{L}^m(D \cap B_\varrho(x)) = \mathcal{L}^m(B_\varrho(x))$ and $\mathcal{L}^m(D \backslash N \cap B_\varrho(x)) < \mathcal{L}^m(B_\varrho(x))$ for all $\varrho \leq \varrho_0$. This implies that $\mathcal{L}^m(N \cap B_\varrho(x)) > 0$, in contradiction to $\mathcal{L}^m(N) = 0$, and hence $(D \backslash N)^1 = D^1$.

Ad 2.): Let $x \in A^1$, then $\lim_{\varrho \to 0} \big(\mathcal{L}^m(A \cap B_\varrho(x)) / \mathcal{L}^m(B_\varrho(x))\big) = 1$. But for all $\varrho > 0$ it holds $\mathcal{L}^m(A \cap B_\varrho(x)) \leq \mathcal{L}^m(B \cap B_\varrho(x))$, which implies that $x \in B^1$. ■

The next theorem, which is due to Federer, states that $\mathfrak{F}E$ is the relevant part of the boundary, since $D \backslash (E^0 \cup \mathfrak{F}E \cup E^1)$ is a $\mathcal{H}^{m-1}$-negligible set.

Theorem A.3 ([AFP05, Th. 3.61] Federer) *Let* E *be a set of finite perimeter in* D. *Then*

$$\mathfrak{F}E \cap D \subset E^{1/2} \subset \partial^* E \quad \text{and} \quad \mathcal{H}^{m-1}(D \backslash (E^0 \cup \mathfrak{F}E \cup E^1)) = 0\,. \tag{A.37}$$

In particular, E *has density either* 0 *or* $1/2$ *or* 1 *at* $\mathcal{H}^{m-1}$*-a.a.* $x \in D$ *and* $\mathcal{H}^{m-1}$*-a.a.* $x \in \partial^* E \cap D$ *belongs to* $\mathfrak{F}E$.

Definition A.12 ([AFP05, Def. 3.63] Approximate limit) *Let $v \in L^1_{\mathrm{loc}}(D)^m$. We say that v has an approximate limit at $x \in D$ if there exists $\bar{v} \in \mathbb{R}^m$ such that*

$$\lim_{\varrho \to 0} \fint_{B_\varrho(x)} |v(y) - \bar{v}| \,\mathrm{d}y = 0\,. \tag{A.38}$$

The set S_v of points where this property does not hold is called the approximate discontinuity set. For any $x \in D \backslash S_v$ the vector $\bar{v}$, uniquely determined by (A.38), is called approximate limit of v at x and denoted by $\tilde{v}(x)$.

We will use the notation

$$B^+_\varrho(x,\nu) := \{y \in B_\varrho(x) \,|\, \langle y - x, \nu\rangle > 0\}, \quad B^-_\varrho(x,\nu) := \{y \in B_\varrho(x) \,|\, \langle y - x, \nu\rangle < 0\}\,.$$

Definition A.13 ([AFP05, Def. 3.67] Approximate jump points) *Let $v \in L^1_{\mathrm{loc}}(D)^m$ and $x \in D$. We say that x is an approximate jump point of v if there exist $a, b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{m-1}$ so that $a \neq b$ and*

$$\lim_{\varrho \to 0} \fint_{B^+_\varrho(x,\nu)} |v(y) - a| \,\mathrm{d}y = 0\,, \quad \lim_{\varrho \to 0} \fint_{B^-_\varrho(x,\nu)} |v(y) - b| \,\mathrm{d}y = 0\,. \tag{A.39}$$

The triple (a, b, ν), uniquely determined by (A.39) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(v^+, v^-, \nu_v(x))$. The set of approximate jump points of v is denoted by J_v.

Definition A.14 ([AFP05, Def. 2.57] Rectifiable sets) *Let $E \subset \mathbb{R}^m$ be an $\mathcal{H}^k$-measurable set. The set E is countably k-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^m$ such that*

$$E \subset \cup_{i=0}^\infty f_i(\mathbb{R}^k)\,; \tag{A.40}$$

E is countably $\mathcal{H}^k$-rectifiable if there are countably many Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^m$ so that

$$\mathcal{H}^k\big(E \backslash \cup_{i=0}^\infty f_i(\mathbb{R}^k)\big) = 0\,. \tag{A.41}$$

Clearly, k-rectifiability implies $\mathcal{H}^k$-rectifiability.

Theorem A.4 ([AFP05, Th. 3.59] De Giorgi) *Let E be an $\mathcal{L}^m$-measurable subset of $\mathbb{R}^m$. Then $\mathfrak{F}E$ is countably $(m-1)$-rectifiable and $|\mathrm{D}\mathcal{X}_E| = \mathcal{H}^{m-1} \lfloor \mathfrak{F}E$.*

By the Besicovitch derivation theorem [AFP05, Th. 2.22] one obtains that for any set of finite perimeter E that $|\mathrm{D}\mathcal{X}_E|$ is concentrated on $\mathfrak{F}E$. Hence, in this case, by Thm. A.4 the Gauss-Green formula (A.31) can be rewritten as

$$\int_E \operatorname{div} \varphi \,\mathrm{d}x = -\int_{\mathfrak{F}E} \nu_E \cdot \varphi \,\mathrm{d}\mathcal{H}^{m-1} \quad \text{for all } \varphi \in C^1_c(D)^m\,. \tag{A.42}$$

Due to Thm. A.4 the perimeter of E can be computed by

$$P(E, D) = \mathcal{H}^{m-1}(D \cap \partial^* E) = \mathcal{H}^{m-1}(D \cap E^{1/2})\,. \tag{A.43}$$

This can be used to rewrite the coarea formula (A.33) using the essential boundary of level sets

$$|\mathrm{D}u|(B) = \int_{-\infty}^{\infty} \mathcal{H}^{m-1}\big(B \cap \partial^*\{u > t\}\big) \,\mathrm{d}t \quad \text{for all Borel sets } B \subset D\,. \tag{A.44}$$

Theorem A.5 ([AFP05, Th. 3.77] Traces on interior rectifiable sets) *Let $v \in BV(D)^m$ and let $\Gamma \subset D$ be a countably $\mathcal{H}^{m-1}$-rectifiable set oriented by ν. Then, for $\mathcal{H}^{m-1}$-a.a. $x \in \Gamma$ there exist $v^+_\Gamma(x)$, $v^-_\Gamma(x) \in \mathbb{R}^m$ such that*

$$\lim_{\varrho \to 0} \fint_{B^+_\varrho(x,\nu(x))} |v(y) - v^+_\Gamma(x)| \,\mathrm{d}y = 0\,, \quad \lim_{\varrho \to 0} \fint_{B^-_\varrho(x,\nu(x))} |v(y) - v^-_\Gamma(x)| \,\mathrm{d}y = 0\,. \tag{A.45}$$

Moreover, $\mathrm{D}v \lfloor \Gamma = (v^+_\Gamma - v^-_\Gamma) \otimes \nu \mathcal{H}^{m-1} \lfloor \Gamma$.

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References

[ADM90] L. Ambrosio and G. Dal Maso. A general chain rule for distributional derivatives. *Proceedings of the American Mathematical Society*, 108(3):691–702, 1990.

[AFP05] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford University Press, 2005.

[Alb00] G. Alberti. Variational models for phase transitions, an approach via Γ-convergence. In *Calculus of variations and partial differential equations (Pisa, 1996)*, pages 95–114. Springer, 2000.

[Att84] H. Attouch. *Variational convergence for functions and operators.* Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[BBR08] E. Bonetti, G. Bonfanti, and R. Rossi. Global existence for a contact problem with adhesion. *Math. Meth. Appl. Sci.*, 31:1029–1064, 2008.

[BBR09] E. Bonetti, G. Bonfanti, and R. Rossi. Thermal effects in adhesive contact: Modelling and analysis. *Nonlinearity*, pages 2697–2731, 2009.

[BFM08] B. Bourdin, G. Francfort, and J.-J. Marigo. The variational approach to fracture. *J. Elasticity*, 91:5–148, 2008.

[BG89] L. Boccardo and T. Gallouët. Non-linear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 1989.

[BSS05] E. Bonetti, G. Schimperna, and A. Segatti. On a doubly nonlinear model for the evolution of damaging in viscoelastic materials. *J. Differential Equations*, 218(1):91–116, 2005.

[DMFT05] G. Dal Maso, G. Francfort, and R. Toader. Quasistatic crack growth in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 176:165–225, 2005.

[EG92] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, 1992.

[Fed69] H. Federer. *Geometric Measure Theory.* Springer, 1969.

[Fei04] E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2004.

[FM06] G. Francfort and A. Mielke. Existence results for a class of rate-independent material models with nonconvex elastic energies. *J. reine angew. Math.*, 595:55–91, 2006.

[FN96] M. Frémond and B. Nedjar. Damage, gradient of damage and principle of virtual power. *Internat. J. Solids Structures*, 33:1083–1103, 1996.

[Fré88] M. Frémond. Contact with adhesion. In J.J. Moreau, P.D. Panagiotopoulos, and G. Strang, editors, *Topics in Nonsmooth Mechanics*, pages 157–186. Birkhäuser, 1988.

[Fré02] M. Frémond. *Non-Smooth Thermomechanics.* Springer-Verlag Berlin Heidelberg, 2002.

[Fun65] Y.C. Fung. *Foundations of solid mechanics.* Prentice-Hall, Inc., 1965.

[Gia05] A. Giacomini. Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fracture. *Calc. Var. Partial Differential Equations*, 22:129–172, 2005.

[Giu84] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Boston, 1984.

[HN75] B. Halphen and Q.S. Nguyen. Sur les matériaux standards généralisés. *J. Mécanique*, 14:39–63, 1975.

[IT79] A. D. Ioffe and V. M. Tihomirov. *Theory of extremal problems*, volume 6 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 1979. Translated from the Russian by Karol Makowski.

[KMR06] M. Kočvara, A. Mielke, and T. Roubíček. A rate-independent approach to the delamination problem. *Math. Mech. Solids*, 11:423–447, 2006.

[KMZ08] D. Knees, A. Mielke, and C. Zanini. On the inviscid limit of a model for crack propagation. *M^3AS*, 18(9):1529–1569, 2008.

[Lew88] J.L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Soc.*, 308(1):177–196, July 1988.

[Mie05] A. Mielke. Evolution in rate-independent systems (Ch.6). In C.M. Dafermos and E. Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations, vol. 2*, pages 461–559. Elsevier B.V., Amsterdam, 2005.

[MM72] M. Marcus and V. J. Mizel. Absolute continuity on tracks and mappings of sobolev spaces. *Arch. Rational Mech. Anal.*, 45:294–320, 1972.

[MM77] L. Modica and S. Mortola. Un esempio di Γ-convergenza. *Boll. U. Mat. Ital. B*, 14(5):285–299, 1977.

[Mod87] L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98(2):123–142, 1987.

[MR06] A. Mielke and T. Roubíček. Rate-independent damage processes in nonlinear elasticity. *M^3AS Math. Models Methods Appl. Sci.*, 16:177–209, 2006.

[MRS08] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Partial Differ. Equ.*, 31:387–416, 2008.

[MRT12] A. Mielke, T. Roubíček, and M. Thomas. From damage to delamination in nonlinearly elastic materials at small strains. *WIAS-Preprint 1542, submitted to J. of Elasticity, accepted*, 2012.

[MT04] A. Mielke and F. Theil. On rate–independent hysteresis models. *Nonl. Diff. Eqns. Appl. (NoDEA)*, 11:151–189, 2004. (Accepted July 2001).

[Nau06] J. Naumann. On the existence of weak solutions to the equations of non-stationary motion of heat-conducting incompressible viscous fluids. *Math. Methods Appl. Sci.*, 29(16):1883–1906, 2006.

[NO08] M. Negri and C. Ortner. Quasi-static crack propagation by griffith's criterion. *M^3AS*, 18(11):1895–1925, 2008.

[Rou05] T. Roubíček. *Nonlinear Partial Differential Equations with Applications.* Birkhäuser, 2005.

[Rou10] T. Roubíček. Thermodynamics of rate-independent processes in viscous solids at small strains. *SIAM J. Math. Anal.*, 42(1):256–297, 2010.

[RR11a] R. Rossi and T. Roubíček. Adhesive contact delaminating at mixed mode, its thermodynamics and analysis. *Preprint arXiv:1110.2794*, submitted, 2011.

[RR11b] R. Rossi and T. Roubíček. Thermodynamics and analysis of rate-independent adhesive contact at small strains. *Nonlinear Anal.*, 74(10):3159–3190, 2011.

[RSZ09] T. Roubíček, L. Scardia, and C. Zanini. Quasistatic delamination problem. *Continuum Mech. Thermodynam.*, 21(3):223–235, 2009.

[Sim87] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl.*, 1987.

[Tho08] M. Thomas. Griffith formula for mode-III-interface-cracks in strainhardening compounds. *Mechanics of Advanced Materials and Structures*, 12(5):428–437, 2008.

[Tho10] M. Thomas. *Rate-independent damage processes in nonlinearly elastic materials.* PhD thesis, Humboldt-Universität zu Berlin, 2010.

[Tho11] M. Thomas. Quasistatic damage evolution with spatial bv-regularization. *WIAS preprint 1638, submitted to DCDS-S, accepted*, 2011.

[TM10] M. Thomas and A. Mielke. Damage of nonlinearly elastic materials at small strain: existence and regularity results. *Zeit. angew. Math. Mech.*, 90(2):88–112, 2010.