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# Nonlocal problems with Neumann boundary conditions

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**Abstract.** We introduce a new Neumann problem for the fractional Laplacian arising from a simple probabilistic consideration, and we discuss the basic properties of this model. We can consider both elliptic and parabolic equations in any domain. In addition, we formulate problems with nonhomogeneous Neumann conditions, and also with mixed Dirichlet and Neumann conditions, all of them having a clear probabilistic interpretation.

We prove that solutions to the fractional heat equation with homogeneous Neumann conditions have the following natural properties: conservation of mass inside $\Omega$, decreasing energy, and convergence to a constant as $t \to \infty$. Moreover, for the elliptic case we give the variational formulation of the problem, and establish existence of solutions.

We also study the limit properties and the boundary behavior induced by this nonlocal Neumann condition.

For concreteness, one may think that our nonlocal analogue of the classical Neumann condition $\partial_\nu u = 0$ on $\partial\Omega$ consists in the nonlocal prescription

$$\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy = 0 \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

## 1. Introduction and results

The aim of this paper is to introduce the following Neumann problem for the fractional Laplacian

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{1.1}$$

Here, $\mathcal{N}_s$ is a new "nonlocal normal derivative", given by

$$\mathcal{N}_s u(x) := c_{n,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy, \qquad x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.2}$$

The normalization constant $c_{n,s}$ is the one appearing in the definition the fractional Laplacian

$$(-\Delta)^s u(x) = c_{n,s}\,\mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. \tag{1.3}$$

See [12, 20] for the basic properties of this operator (and for further details on the normalization constant $c_{n,s}$, whose explicit value only plays a minor role in this paper).

As we will see below, the corresponding heat equation with homogeneous Neumann conditions

$$\begin{cases} u_t + (-\Delta)^s u &= 0 \quad \text{in } \Omega, \quad t>0 \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \quad t>0 \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \quad t=0 \end{cases} \tag{1.4}$$

possesses natural properties like conservation of mass inside $\Omega$ or convergence to a constant as $t \to +\infty$ (see Section 4).

The probabilistic interpretation of the Neumann problem (1.4) may be summarized as follows:

(1) $u(x,t)$ is the probability distribution of the position of a particle moving randomly inside $\Omega$.

(2) When the particle exits $\Omega$, it immediately comes back into $\Omega$.

(3) The way in which it comes back inside $\Omega$ is the following: If the particle has gone to $x \in \mathbb{R}^n \setminus \overline{\Omega}$, it may come back to any point $y \in \Omega$, the probability of jumping from $x$ to $y$ being proportional to $|x-y|^{-n-2s}$.

These three properties lead to the equation (1.4), being $u_0$ the initial probability distribution of the position of the particle.

A variation of formula (1.2) consists in renormalizing $\mathcal{N}_s u$ according to the underlying probability law induced by the Lévy process. This leads to the definition

$$\tilde{\mathcal{N}_s} u(x) := \frac{\mathcal{N}_s u(x)}{c_{n,s} \int_\Omega \frac{dy}{|x-y|^{n+2s}}}. \tag{1.5}$$

Other Neumann problems for the fractional Laplacian (or other nonlocal operators) were introduced in [4, 8], [1, 3], [9, 10, 11], and [15]. All these different Neumann problems for nonlocal operators recover the classical Neumann problem as a limit case, and most of them has clear probabilistic interpretations as well. We postpone to Section 7 a comparison between these different models and ours.

An advantage of our approach is that the problem has a variational structure. In particular, we show that the classical integration by parts formulae

$$\int_\Omega \Delta u = \int_{\partial\Omega} \partial_\nu u$$

$$\text{and} \qquad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega v\,(-\Delta) u + \int_{\partial\Omega} v \partial_\nu u$$

are replaced in our setting by

$$\int_\Omega (-\Delta)^s u \, dx = -\int_{\mathbb{R}^n\setminus\Omega} \mathcal{N}_s u \, dx$$

and

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}} \, dx\, dy = \int_\Omega v\,(-\Delta)^s u + \int_{\mathbb{R}^n\setminus\Omega} v\, \mathcal{N}_s u.$$

Also, the classical Neumann problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \partial_\nu u &= g \quad \text{on } \partial\Omega \end{cases} \tag{1.6}$$

comes from critical points of the energy functional

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u - \int_{\partial\Omega} g\, u,$$

without trace conditions. In analogy with this, we show that our nonlocal Neumann condition

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ \mathcal{N}_s u &= g \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega} \end{cases} \tag{1.7}$$

follows from free critical points of the energy functional

$$\frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dx\, dy - \int_\Omega f\, u - \int_{\mathbb{R}^n\setminus\Omega} g\, u,$$

see Proposition 3.7. Moreover, as well known, the theory of existence and uniqueness of solutions for the classical Neumann problem (1.6) relies on the compatibility condition

$$\int_\Omega f = -\int_{\partial\Omega} g.$$

We provide the analogue of this compatibility condition in our framework, that is

$$\int_\Omega f = -\int_{\mathbb{R}^n\setminus\Omega} g,$$

see Theorem 3.9. Also, we give a description of the spectral properties of our nonlocal problem, which are in analogy with the classical case.

The paper is organized in this way. In Section 2 we give a probabilistic interpretation of our Neumann condition, as a random reflection of a particle inside the domain, according to a Lévy flight. This also allows us to consider mixed Dirichlet and Neumann conditions and to get a suitable heat equation from the stochastic process.

In Section 3 we consider the variational structure of the associated nonlocal elliptic problem, we show an existence and uniqueness result (namely Theorem 3.9), as follows:

*Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $f \in L^2(\Omega)$, and $g \in L^1(\mathbb{R}^n \setminus \Omega)$. Suppose that there exists a $C^2$ function $\psi$ such that $\mathcal{N}_s\psi = g$ in $\mathbb{R}^n \setminus \overline{\Omega}$.*

*Then, problem* (3.9) *admits a weak solution if and only if*

$$\int_\Omega f = -\int_{\mathbb{R}^n\setminus\Omega} g.$$

*Moreover, if such a compatibility condition holds, the solution is unique up to an additive constant.*

Also, we give a description of the eigenvalues of $(-\Delta)^s$ with zero Neumann boundary conditions (see Theorem 3.11):

*Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and consider the eigenvalue problem*

$$\begin{cases} (-\Delta)^s u &= \lambda u \quad \text{in } \Omega \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

*Then, there exists a sequence of nonnegative eigenvalues*

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots,$$

*and its corresponding eigenfunctions are a complete orthogonal system in $L^2(\Omega)$.*

In Section 4 we discuss the associated heat equation. As it happens in the classical case, we show that such equation preserves the mass, it has decreasing energy, and the solutions approach a constant as $t \to +\infty$. In particular, by the results in Propositions 4.1, 4.2 and 4.3 we have:

*Assume that $u(x,t)$ is a classical solution to*

$$\begin{cases} u_t + (-\Delta)^s u &= 0 \quad \text{in } \Omega, \quad t > 0 \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \quad t > 0 \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \quad t = 0. \end{cases}$$

*Then the total mass is conserved, i.e. for all $t > 0$*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)dx.$$

*Moreover, the energy*

$$E(t) = \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{n+2s}}\,dx\,dy$$

*is decreasing in time $t > 0$.*

*Finally, the solution approaches a constant for large times: more precisely*

$$u \longrightarrow \frac{1}{|\Omega|}\int_\Omega u_0 \quad \text{in } L^2(\Omega)$$

*as $t \to +\infty$.*

In Section 5 we compute some limits when $s \to 1$, showing that we can recover the classical case. In particular, we show in Proposition 5.1 that:

*Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain. Let $u$ and $v$ be $C_0^2(\mathbb{R}^n)$ functions. Then,*

$$\lim_{s\to 1}\int_{\mathbb{R}^n\setminus\Omega} \mathcal{N}_s u\, v = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, v.$$

Also, we prove that nice functions can be extended continuously outside $\overline{\Omega}$ in order to satisfy a homogeneous nonlocal Neumann condition, and we characterize the boundary behavior of the nonlocal Neumann function. More precisely, in Proposition 5.2 we show that:

*Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^1$ boundary. Let $u$ be continuous in $\overline{\Omega}$, with $\mathcal{N}_s u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Then $u$ is continuous in the whole of $\mathbb{R}^n$.*

The boundary behavior of the nonolcal Neumann condition is also addressed in Proposition 5.4:

*Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ domain, and $u \in C(\mathbb{R}^n)$. Then, for all $s \in (0,1)$,*

$$\lim_{\substack{x\to\partial\Omega \\ x\in\mathbb{R}^n\setminus\overline{\Omega}}} \tilde{\mathcal{N}}_s u(x) = 0,$$

*where $\tilde{\mathcal{N}}$ is defined by (1.5).*

*Also, if $s > \frac{1}{2}$ and $u \in C^{1,\alpha}(\mathbb{R}^n)$ for some $\alpha > 0$, then*

$$\partial_\nu \tilde{\mathcal{N}}_s u(x) := \lim_{\epsilon\to 0^+} \frac{\tilde{\mathcal{N}}_s u(x+\epsilon\nu)}{\epsilon} = \kappa\,\partial_\nu u \qquad \text{for any } x \in \partial\Omega,$$

*for some constant $\kappa > 0$.*

Later on, in Section 6 we deal with an overdetermined problem and we show that it is not possible to prescribe both nonlocal Neumann and Dirichlet conditions for a continuous function.

Finally, in Section 7 we recall the various nonlocal Neumann conditions already appeared in the literature, and we compare them with our model.

All the arguments presented are of elementary nature. Moreover, all our considerations work for any general Lévy measure $\nu$ satisfying

$$\int_{\mathbb{R}^n} \min(1, |y|^2) d\nu(y) < +\infty.$$

However, for sake of clarity of presentation, we have done everything for the most canonical case of the fractional Laplacian.

## 2. Heuristic probabilistic interpretation

Let us consider the Lévy process in $\mathbb{R}^n$ whose infinitesimal generator is the fractional Laplacian $(-\Delta)^s$. Heuristically, we may think that this process represents the (random) movement of a particle along time $t > 0$. As it is well known, the probability density $u(x,t)$ of the position of the particle solves the fractional heat equation $u_t + (-\Delta)^s u = 0$ in $\mathbb{R}^n$; see [21] for a simple illustration of this fact.

Recall that when the particle is situated at $x \in \mathbb{R}^n$, it may jump to any other point $y \in \mathbb{R}^n$, the probability of jumping to $y$ being proportional to $|x-y|^{-n-2s}$.

Similarly, one may consider the random movement of a particle inside a bounded domain $\Omega \subset \mathbb{R}^n$, but in this case one has to decide what happens when the particle leaves $\Omega$.

In the classical case $s = 1$ (when the Lévy process is the Brownian motion), we have the following:

(1) If the particle is *killed* when it reaches the boundary $\partial\Omega$, then the probability distribution solves the heat equation with homogeneous *Dirichlet* conditions.
(2) If, instead, when the particle reaches the boundary $\partial\Omega$ it immediately *comes back into* $\Omega$ (i.e., it bounces on $\partial\Omega$), then the probability distribution solves the heat equation with homogeneous *Neumann* conditions.

In the nonlocal case $s \in (0,1)$, in which the process has jumps, case (1) corresponds to the following: The particle is killed when it exits $\Omega$. In this case, the probability distribution $u$ of the process solves the heat equation with homogeneous Dirichlet conditions $u = 0$ in $\mathbb{R}^n \setminus \Omega$, and solutions to this problem are well understood; see for example [18, 14, 13, 2].

The analogue of case (2) is the following: When the particle exits $\Omega$, it immediately comes back into $\Omega$. Of course, one has to decide how the particle comes back into the domain.

In [1, 3] the idea was to find a deterministic "reflection" or "projection" which describes the way in which the particle comes back into $\Omega$.

The alternative that we propose here is the following: If the particle has gone to $x \in \mathbb{R}^n \setminus \overline{\Omega}$, then it may come back to *any* point $y \in \Omega$, the probability of jumping from $x$ to $y$ being proportional to $|x-y|^{-n-2s}$.

Notice that this is exactly the (random) way as the particle is moving all the time, here we just add the restriction that it has to immediately come back into $\Omega$ every time it goes outside.

Let us finally illustrate how this random process leads to problems (1.1) or (1.4). In fact, to make the exposition easier, we will explain the case of mixed Neumann and Dirichlet conditions, which, we think, is very natural.

2.1. **Mixed Dirichlet and Neumann conditions.** Assume that we have some domain $\Omega \subset \mathbb{R}^n$, and that its complement $\mathbb{R}^n \setminus \overline{\Omega}$ is splitted into two parts: $N$ (with Neumann conditions), and $D$ (with Dirichlet conditions).

Consider a particle moving randomly, starting inside $\Omega$. When the particle reaches $D$, it obtains a payoff $\phi(x)$, which depends on the point $x \in D$ where the particle arrived. Instead, when the particle reaches $N$ it immediately comes back to $\Omega$ as described before.

If we denote $u(x)$ the expected payoff, then we clearly have

$$(-\Delta)^s u = 0 \quad \text{in } \Omega$$

and

$$u = \phi \quad \text{in } D,$$

where $\phi : D \longrightarrow \mathbb{R}$ is a given function.

Moreover, recall that when the particle is in $x \in N$ then it goes back to some point $y \in \Omega$, with probability proportional to $|x-y|^{-n-2s}$. Hence, we have that

$$u(x) = \kappa \int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\, dy \qquad \text{for } x \in N,$$

for some constant $\kappa$, possibly depending on the point $x \in N$, that has been fixed. In order to normalize the probability measure, the value of the constant $\kappa$ is so that

$$\kappa \int_\Omega \frac{dy}{|x-y|^{n+2s}} = 1.$$

Finally, the previous identity can be written as

$$\mathcal{N}_s u(x) = c_{n,s} \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy = 0 \quad \text{for } x \in N,$$

and therefore $u$ solves

$$\left\{ \begin{array}{rcll} (-\Delta)^s u &=& 0 & \text{in } \Omega \\ \mathcal{N}_s u &=& 0 & \text{in } N \\ u &=& \phi & \text{in } D, \end{array} \right.$$

which is a nonlocal problem with mixed Neumann and Dirichlet conditions.

Note that the previous problem is the nonlocal analogue of

$$\left\{ \begin{array}{rcll} -\Delta u &=& 0 & \text{in } \Omega \\ \partial_\nu u &=& 0 & \text{in } \Gamma_N \\ u &=& \phi & \text{in } \Gamma_D, \end{array} \right.$$

being $\Gamma_D$ and $\Gamma_N$ two disjoint subsets of $\partial\Omega$, in which classical Dirichlet and Neumann boundary conditions are prescribed.

More generally, the classical Robin condition $a\partial_\nu u + bu = c$ on some $\Gamma_R \subseteq \partial\Omega$ may be replaced in our nonlocal framework by $a\mathcal{N}_s u + bu = c$ on some $R \subseteq \mathbb{R}^n \setminus \overline{\Omega}$. Nonlinear boundary conditions may be considered in a similar way.

2.2. **Fractional heat equation, nonhomogeneous Neumann conditions.** Let us consider now the random movement of the particle inside $\Omega$, with our new Neumann conditions in $\mathbb{R}^n \setminus \overline{\Omega}$.

Denoting $u(x,t)$ the probability density of the position of the particle at time $t > 0$, with a similar discretization argument as in [21], one can see that $u$ solves the fractional heat equation

$$u_t + (-\Delta)^s u = 0 \quad \text{in } \Omega \quad \text{for } t > 0,$$

with

$$\mathcal{N}_s u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega} \quad \text{for } t > 0.$$

Thus, if $u_0$ is the initial probability density, then $u$ solves problem (1.4).

Of course, one can now see that with this probabilistic interpretation there is no problem in considering a right hand side $f$ or nonhomogeneous Neumann conditions

$$\begin{cases} u_t + (-\Delta)^s u &= f(x,t,u) \quad \text{in } \Omega \\ \mathcal{N}_s u &= g(x,t) \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

In this case, $g$ represents a "nonlocal flux" of new particles coming from outside $\Omega$, and $f$ would represent a reaction term.

## 3. The elliptic problem

Given $g \in L^1(\mathbb{R}^n \setminus \Omega)$ and measurable functions $u, v : \mathbb{R}^n \to \mathbb{R}$, we set

$$\|u\|_{H^s_\Omega} := \sqrt{\|u\|^2_{L^2(\Omega)} + \||g|^{1/2}\, u\|^2_{L^2(\mathbb{R}^n\setminus\Omega)} + \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy} \tag{3.1}$$

and

$$\begin{aligned} (u,v)_{H^s_\Omega} := &\int_\Omega u\, v\, dx + \int_{\mathbb{R}^n\setminus\Omega} |g|\, u\, v\, dx \\ &+ \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy. \end{aligned} \tag{3.2}$$

Then, we define the space

$$H^s_\Omega := \left\{ u : \mathbb{R}^n \to \mathbb{R} \ \text{ measurable } \ : \|u\|_{H^s_\Omega} < +\infty \right\}.$$

Notice that $H^s_\Omega$ and its norm depend on $g$, but we omit this dependence in the notation for the sake of simplicity.

**Proposition 3.1.** *$H^s_\Omega$ is a Hilbert space with the scalar product defined in* (3.2).

*Proof.* We point out that (3.2) is a bilinear form and $\|u\|_{H^s_\Omega} = \sqrt{(u,u)_{H^s_\Omega}}$. Also, if $\|u\|_{H^s_\Omega} = 0$, it follows that $\|u\|_{L^2(\Omega)} = 0$, hence $u = 0$ a.e. in $\Omega$, and that

$$\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = 0,$$

which in turn implies that $|u(x) - u(y)| = 0$ for any $(x,y) \in \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$. In particular, a.e. $x \in \mathcal{C}\Omega$ and $y \in \Omega$ we have that

$$u(x) = u(x) - u(y) = 0.$$

This shows that $u = 0$ a.e. in $\mathbb{R}^n$, so it remains to prove that $H^s_\Omega$ is complete. For this, we take a Cauchy sequence $u_k$ with respect to the norm in (3.1).

In particular, $u_k$ is a Cauchy sequence in $L^2(\Omega)$ and therefore, up to a subsequence, we suppose that $u_k$ converges to some $u$ in $L^2(\Omega)$ and a.e. in $\Omega$. More explicitly, there exists $Z_1 \subset \mathbb{R}^n$ such that

$$|Z_1| = 0 \text{ and } u_k(x) \to u(x) \text{ for every } x \in \Omega \setminus Z_1. \tag{3.3}$$

Also, given any $U : \mathbb{R}^n \to \mathbb{R}$, for any $(x, y) \in \mathbb{R}^{2n}$ we define

$$E_U(x,y) := \frac{\Big(U(x) - U(y)\Big)\, \chi_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2}(x,y)}{|x-y|^{\frac{n+2s}{2}}}. \tag{3.4}$$

Notice that

$$E_{u_k}(x,y) - E_{u_h}(x,y) = \frac{\Big(u_k(x) - u_h(x) - u_k(y) + u_h(y)\Big)\, \chi_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2}(x,y)}{|x-y|^{\frac{n+2s}{2}}}.$$

Accordingly, since $u_k$ is a Cauchy sequence in $H^s_\Omega$, for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that if $h,\, k \geq N_\epsilon$ then

$$\epsilon^2 \geq \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|(u_k - u_h)(x) - (u_k - u_h)(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \|E_{u_k} - E_{u_h}\|^2_{L^2(\mathbb{R}^{2n})}.$$

That is, $E_{u_k}$ is a Cauchy sequence in $L^2(\mathbb{R}^{2n})$ and thus, up to a subsequence, we assume that $E_{u_k}$ converges to some $E$ in $L^2(\mathbb{R}^{2n})$ and a.e. in $\mathbb{R}^{2n}$. More explicitly, there exists $Z_2 \subset \mathbb{R}^{2n}$ such that

$$|Z_2| = 0 \text{ and } E_{u_k}(x,y) \to E(x,y) \text{ for every } (x,y) \in \mathbb{R}^{2n} \setminus Z_2. \tag{3.5}$$

Now, for any $x \in \Omega$, we set

$$\begin{aligned} & S_x := \{y \in \mathbb{R}^n \,:\, (x,y) \in \mathbb{R}^{2n} \setminus Z_2\}, \\ & W := \{(x,y) \in \mathbb{R}^{2n} \,:\, x \in \Omega \text{ and } y \in \mathbb{R}^n \setminus S_x\} \\ \text{and} \qquad & V := \{x \in \Omega \,:\, |\mathbb{R}^n \setminus S_x| = 0\}. \end{aligned}$$

We remark that

$$W \subseteq Z_2. \tag{3.6}$$

Indeed, if $(x, y) \in W$ we have that $y \in \mathbb{R}^n \setminus S_x$, that is $(x, y) \notin \mathbb{R}^{2n} \setminus Z_2$, and so $(x, y) \in Z_2$, which gives (3.6).

Using (3.5) and (3.6), we obtain that $|W| = 0$, hence by the Fubini's Theorem we have that

$$0 = |W| = \int_\Omega |\mathbb{R}^n \setminus S_x|\, dx,$$

which implies that $|\mathbb{R}^n \setminus S_x| = 0$ for a.e. $x \in \Omega$.

As a consequence, we conclude that $|\Omega \setminus V| = 0$. This and (3.3) imply that

$$|\Omega \setminus (V \setminus Z_1)| = |(\Omega \setminus V) \cup Z_1| \leq |\Omega \setminus V| + |Z_1| = 0.$$

In particular, we have that $V \setminus Z_1 \neq \varnothing$, so we can fix $x_0 \in V \setminus Z_1$.

Since $x_0 \in \Omega \setminus Z_1$, we know from (3.3) that

$$\lim_{k\to+\infty} u_k(x_0) = u(x_0).$$

Furthermore, since $x_0 \in V$ we have that $|\mathbb{R}^n \setminus S_{x_0}| = 0$. That is, a.e. $y \in \mathbb{R}^n$ (namely, for every $y \in S_{x_0}$), we have that $(x_0, y) \in \mathbb{R}^{2n} \setminus Z_2$ and so

$$\lim_{k\to+\infty} E_{u_k}(x_0, y) = E(x_0, y),$$

thanks to (3.5). Notice also that $\Omega \times (\mathcal{C}\Omega) \subseteq \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$ and so, recalling (3.4), we have that

$$E_{u_k}(x_0, y) := \frac{u_k(x_0) - u_k(y)}{|x_0 - y|^{\frac{n+2s}{2}}},$$

for a.e. $y \in \mathcal{C}\Omega$. Thus, we obtain

$$\begin{aligned} \lim_{k\to+\infty} u_k(y) &= \lim_{k\to+\infty} \left\{ u_k(x_0) - |x_0 - y|^{\frac{n+2s}{2}} E_{u_k}(x_0, y) \right\} \\ &= u(x_0) - |x_0 - y|^{\frac{n+2s}{2}} E(x_0, y), \end{aligned}$$

a.e. $y \in \mathcal{C}\Omega$.

This and (3.3) say that $u_k$ converges a.e. in $\mathbb{R}^n$. Up to a change of notation, we will say that $u_k$ converges a.e. in $\mathbb{R}^n$ to some $u$. So, using that $u_k$ is a Cauchy sequence in $H^s_\Omega$, fixed any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that, for any $h \geq N_\epsilon$,

$$\begin{aligned} \epsilon^2 &\geq \liminf_{k\to+\infty} \|u_h - u_k\|^2_{H^s_\Omega} \\ &\geq \liminf_{k\to+\infty} \int_\Omega (u_h - u_k)^2 + \liminf_{k\to+\infty} \int_{\mathcal{C}\Omega} |g|(u_h - u_k)^2 \\ &\qquad + \liminf_{k\to+\infty} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|(u_h - u_k)(x) - (u_h - u_k)(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \\ &\geq \int_\Omega (u_h - u)^2 + \int_{\mathcal{C}\Omega} |g|(u_h - u)^2 + \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|(u_h - u)(x) - (u_h - u)(y)|^2}{|x - y|^{n+2s}}\, dx\, dy \\ &= \|u_h - u\|^2_{H^s_\Omega}, \end{aligned}$$

where Fatou's Lemma was used. This says that $u_h$ converges to $u$ in $H^s_\Omega$, showing that $H^s_\Omega$ is complete. ☐

3.1. **Some integration by parts formulas.** The following is a nonlocal analogue of the divergence theorem.

**Lemma 3.2.** *Let $u$ be any bounded $C^2$ function in $\mathbb{R}^n$. Then,*

$$\int_\Omega (-\Delta)^s u = -\int_{\mathbb{R}^n\setminus\Omega} \mathcal{N}_s u.$$

*Proof.* Note that

$$\int_\Omega \int_\Omega \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dx\, dy = \int_\Omega \int_\Omega \frac{u(y) - u(x)}{|x - y|^{n+2s}}\, dx\, dy = 0,$$

since the role of $x$ and $y$ in the integrals above is symmetric. Hence, we have that

$$\begin{aligned} \int_\Omega (-\Delta)^s u\, dx &= c_{n,s} \int_\Omega \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy\, dx = c_{n,s} \int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy\, dx \\ &= c_{n,s} \int_{\mathbb{R}^n\setminus\Omega} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dx\, dy = -\int_{\mathbb{R}^n\setminus\Omega} \mathcal{N}_s u(y)\, dy, \end{aligned}$$

as desired. □

More generally, we have the following integration by parts formula.

**Lemma 3.3.** *Let $u$ and $v$ be bounded $C^2$ functions in $\mathbb{R}^n$. Then,*

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2}\frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy=\int_\Omega v\,(-\Delta)^s u+\int_{\mathbb{R}^n\setminus\Omega} v\,\mathcal{N}_s u\,,$$

*where $c_{n,s}$ is the constant in* (1.3).

*Proof.* Notice that

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2}\frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy\\&\quad=\int_\Omega\int_{\mathbb{R}^n} v(x)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\,dx+\int_{\mathbb{R}^n\setminus\Omega}\int_\Omega v(x)\,\frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy\,dx.\end{aligned}$$

Thus, using (1.3) and (1.2), the identity follows. □

*Remark* 3.4. We recall that if one takes $\partial_\nu u=1$, then one can obtain the perimeter of $\Omega$ by integrating this Neumann condition over $\partial\Omega$. Indeed,

$$|\partial\Omega|=\int_{\partial\Omega}dx=\int_{\partial\Omega}\partial_\nu u\,dx. \tag{3.7}$$

Analogously, we can define $\tilde{\mathcal{N}_s}u$, by renormalizing $\mathcal{N}_s u$ by a factor

$$w_{s,\Omega}(x):=c_{n,s}\int_\Omega\frac{dy}{|x-y|^{n+2s}},$$

that is

$$\tilde{\mathcal{N}_s}u(x):=\frac{\mathcal{N}_s u(x)}{w_{s,\Omega}(x)}\quad\text{for}\quad x\in\mathbb{R}^n\setminus\overline{\Omega}. \tag{3.8}$$

Now, we observe that if $\tilde{\mathcal{N}_s}u(x)=1$ for any $x\in\mathbb{R}^n\setminus\overline{\Omega}$, then we find the fractional perimeter (see [6] where this object was introduced) by integrating such nonlocal Neumann condition over $\mathbb{R}^n\setminus\Omega$, that is:

$$\begin{aligned}\mathrm{Per}_s(\Omega)&:=c_{n,s}\int_\Omega\int_{\mathbb{R}^n\setminus\Omega}\frac{dx\,dy}{|x-y|^{n+2s}}\\&=\int_{\mathbb{R}^n\setminus\Omega}w_{s,\Omega}(x)\,dx\\&=\int_{\mathbb{R}^n\setminus\Omega}w_{s,\Omega}(x)\,\tilde{\mathcal{N}_s}u(x)\,dx\\&=\int_{\mathbb{R}^n\setminus\Omega}\mathcal{N}_s u(x)\,dx,\end{aligned}$$

that can be seen as the nonlocal counterpart of (3.7).

*Remark* 3.5. The renormalized Neumann condition in (3.8) can also be framed into the probabilistic interpretation of Section 2.

Indeed suppose that $\mathcal{C}\Omega$ is partitioned into a Dirichlet part $D$ and a Neumann part $N$ and that:

- our Lévy process receives a final payoff $\phi(x)$ when it leaves the domain $\Omega$ by landing at the point $x$ in $D$,
- if the Lévy process leaves $\Omega$ by landing at the point $x$ in $N$, then it receives an additional payoff $\psi(x)$ and is forced to come back to $\Omega$ and keep running by following the same probability law (the case discussed in Section 2 is the model situation in which $\psi \equiv 0$).

In this setting, the expected payoff $u(x)$ obtained by starting the process at the point $x \in \Omega$ satisfies $(-\Delta)^s u = 0$ in $\Omega$ and $u = \phi$ in $D$. Also, for any $x \in N$, the expected payoff landing at $x$ must be equal to the additional payoff $\psi(x)$ plus the average payoff $u(y)$ obtained by jumping from $x$ to $y \in \Omega$, that is:

$$\text{for any } x \in N, \qquad u(x) = \psi(x) + \frac{\displaystyle\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}},$$

which corresponds to $\tilde{\mathcal{N}}_s u(x) = \psi(x)$.

3.2. **Weak solutions with Neumann conditions.** The integration by parts formula from Lemma 3.3 leads to the following:

**Definition 3.6.** Let $f \in L^2(\Omega)$ and $g \in L^1(\mathbb{R}^n \setminus \Omega)$. Let $u \in H^s_\Omega$. We say that $u$ is a weak solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ \mathcal{N}_s u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \end{cases} \tag{3.9}$$

whenever

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega f\,v + \int_{\mathbb{R}^n\setminus\Omega} g\,v \tag{3.10}$$

for all test functions $v \in H^s_\Omega$.

With this definition, we can prove the following.

**Proposition 3.7.** *Let $f \in L^2(\Omega)$ and $g \in L^1(\mathbb{R}^n \setminus \Omega)$. Let $I : H^s_\Omega \to \mathbb{R}$ be the functional defined as*

$$I[u] := \frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega f\,u - \int_{\mathbb{R}^n\setminus\Omega} g\,u$$

*for every $u \in H^s_\Omega$.*

*Then any critical point of $I$ is a weak solution of* (3.9).

*Proof.* First of all, we observe that the functional $I$ is well defined on $H^s_\Omega$. Indeed, if $u \in H^s_\Omega$ then

$$\left| \int_\Omega f\,u \right| \le \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \le C\,\|u\|_{H^s_\Omega},$$

and

$$\left| \int_{\mathbb{R}^n\setminus\Omega} g\,u \right| \le \int_{\mathbb{R}^n\setminus\Omega} |g|^{1/2}\,|g|^{1/2}\,|u| \le \|g\|^{1/2}_{L^1(\mathbb{R}^n\setminus\Omega)}\,\||g|^{1/2}u\|_{L^2(\mathbb{R}^n\setminus\Omega)} \le C\,\|u\|_{H^s_\Omega}.$$

Therefore, if $u \in H^s_\Omega$ we have that

$$|I[u]| \leq C\|u\|_{H^s_\Omega} < +\infty.$$

Now, we compute the first variation of $I$. For this, we take $|\epsilon| < 1$ and $v \in H^s_\Omega$. Then the function $u + \epsilon v \in H^s_\Omega$, and so we can compute

$$\begin{aligned} I[u+\epsilon v] &= \frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|(u+\epsilon v)(x) - (u+\epsilon v)(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad - \int_\Omega f(u+\epsilon v) - \int_{\mathbb{R}^n\setminus\Omega} g(u+\epsilon v) \\ &= I(u) + \epsilon \left( \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega f\, v - \int_{\mathbb{R}^n\setminus\Omega} g\, v \right) \\ &\quad + \frac{c_{n,s}}{4}\epsilon^2 \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy. \end{aligned}$$

Hence,

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{I[u+\epsilon v] - I[u]}{\epsilon} \\ &\qquad = \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega f\, v - \int_{\mathbb{R}^n\setminus\Omega} g\, v, \end{aligned}$$

which means that

$$I'[u](v) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy - \int_\Omega f\, v - \int_{\mathbb{R}^n\setminus\Omega} g\, v.$$

Therefore, if $u$ is a critical point of $I$, then $u$ is a weak solution to (3.9), according to Definition 3.6. □

Next result is a sort of maximum principle and it is auxiliary towards the existence and uniqueness theory provided in the subsequent Theorem 3.9.

**Lemma 3.8.** *Let $f \in L^2(\Omega)$ and $g \in L^1(\mathbb{R}^n \setminus \Omega)$. Let $u$ be any $H^s_\Omega$ function satisfying in the weak sense*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ \mathcal{N}_s u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

*with $f \geq 0$ and $g \geq 0$.*

*Then, $u$ is constant.*

*Proof.* First, we observe that the function $v \equiv 1$ belongs to $H^s_\Omega$, and therefore we can use it as a test function in (3.10), obtaining that

$$0 \leq \int_\Omega f = -\int_{\mathbb{R}^n\setminus\Omega} g \leq 0.$$

This implies that

$$f = 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad g = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega.$$

Therefore, taking $v = u$ as a test function in (3.10), we deduce that

$$\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0,$$

and hence $u$ must be constant. □

We can now give the following existence and uniqueness result (we observe that its statement is in complete analogy[1] with the classical case, see e.g. page 294 in [16]).

**Theorem 3.9.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $f \in L^2(\Omega)$, and $g \in L^1(\mathbb{R}^n \setminus \Omega)$. Suppose that there exists a $C^2$ function $\psi$ such that $\mathcal{N}_s\psi = g$ in $\mathbb{R}^n \setminus \overline{\Omega}$.*

*Then, problem* (3.9) *admits a solution in $H^s_\Omega$ if and only if*

$$\int_\Omega f = -\int_{\mathbb{R}^n\setminus\Omega} g. \tag{3.11}$$

*Moreover, in case that* (3.11) *holds, the solution is unique up to an additive constant.*

*Proof. Case 1.* We do first the case $g \equiv 0$, i.e., with homogeneous nonlocal Neumann conditions. We also assume that $f \not\equiv 0$, otherwise there is nothing to prove.

Given $h \in L^2(\Omega)$, we look for a solution $v \in H^s_\Omega$ of the problem

$$\int_\Omega v\,\varphi + \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega h\,\varphi, \tag{3.12}$$

for any $\varphi \in H^s_\Omega$, with homogeneous Neumann conditions $\mathcal{N}_s v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

We consider the functional $\mathcal{F} : H^s_\Omega \to \mathbb{R}$ defined as

$$\mathcal{F}(\varphi) := \int_\Omega h\,\varphi \quad \text{for any } \varphi \in H^s_\Omega.$$

It is easy to see that $\mathcal{F}$ is linear. Moreover, it is continuous on $H^s_\Omega$:

$$|\mathcal{F}(\varphi)| \le \int_\Omega |h|\,|\varphi| \le \|h\|_{L^2(\Omega)}\,\|\varphi\|_{L^2(\Omega)} \le \|h\|_{L^2(\Omega)}\,\|\varphi\|_{H^s_\Omega}.$$

Therefore, from the Riesz representation theorem it follows that problem (3.12) admits a unique solution $v \in H^s_\Omega$ for any given $h \in L^2(\Omega)$.

Furthermore, taking $\varphi := v$ in (3.12), one obtain that

$$\|v\|_{H^s(\Omega)} \le C\|h\|_{L^2(\Omega)}. \tag{3.13}$$

Now, we define he operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ as $Th = v$. We have that $T$ is compact. Indeed, we take a sequence $\{h_k\}_{k\in\mathbb{N}}$ bounded in $L^2(\Omega)$. Hence, from (3.13) we deduce that the sequence of $v_k := Th_k$ is bounded in $H^s(\Omega)$, which is compactly embedded in $L^2(\Omega)$ (see e.g. [12]). Therefore, there exists a subsequence that converges in $L^2(\Omega)$.

Now, we show that $T$ is self-adjoint. For this, we take $h_1, h_2 \in C^\infty_0(\Omega)$ and we use the weak formulation in (3.12) to say that, for every $\varphi, \phi \in H^s_\Omega$, we have

$$\int_\Omega Th_1\,\varphi + \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(Th_1(x)-Th_1(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega h_1\,\varphi, \tag{3.14}$$

[1]The only difference with the classical case is that in Theorem 3.9 it is *not* necessary to suppose that the domain is connected in order to obtain the uniqueness result.

and

$$\int_\Omega Th_2\,\phi + \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{(Th_2(x)-Th_2(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega h_2\,\phi, \tag{3.15}$$

Now we take $\varphi := Th_2$ and $\phi := Th_1$ in (3.14) and (3.15) respectively and we obtain that

$$\int_\Omega h_1\,Th_2 = \int_\Omega Th_1\,h_2 \tag{3.16}$$

for any $h_1, h_2 \in C_0^\infty(\Omega)$. If $h_1, h_2 \in L^2(\Omega)$, there exist sequences of functions in $C_0^\infty(\Omega)$, say $h_{1,k}$ and $h_{2,k}$, such that $h_{1,k} \to h_1$ and $h_{2,k} \to h_2$ in $L^2(\Omega)$ as $k \to +\infty$. From (3.16) we have that

$$\int_\Omega h_{1,k}\,Th_{2,k} = \int_\Omega Th_{1,k}\,h_{2,k}. \tag{3.17}$$

Moreover, from (3.13) we deduce that $Th_{1,k} \to Th_1$ and $Th_{2,k} \to Th_2$ in $H^s(\Omega)$ as $k \to +\infty$, and so

$$\int_\Omega h_{1,k}\,Th_{2,k} \to \int_\Omega h_1\,Th_2 \quad \text{as } k \to +\infty$$

and

$$\int_\Omega Th_{1,k}\,h_{2,k} \to \int_\Omega Th_1\,h_2 \quad \text{as } k \to +\infty.$$

The last two formulas and (3.17) imply that

$$\int_\Omega h_1\,Th_2 = \int_\Omega Th_1\,h_2 \quad \text{for any } h_1, h_2 \in L^2(\Omega),$$

which says that $T$ is self-adjoint.

Let us now come back to the equation $(-\Delta)^s u = f$ in $\Omega$, $\mathcal{N}_s u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. This equation can be written in terms of the operator $T$ as follows

$$(Id - T)^{-1} f = w,$$

with $w = f + u$. Therefore, by the Fredholm Alternative, the equation admits a solution if and only if $f \in \mathrm{Ker}(I-T)^\perp$. But

$$\mathrm{Ker}(I-T) = \left\{ u \in H^s_\Omega \,:\, (-\Delta)^s u = 0 \text{ in } \Omega,\ \mathcal{N}_s u = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega} \right\},$$

which by Lemma 3.8 consists only of constant functions.

Thus, it follows that the equation admits a solution if and only if $\int_\Omega f = 0$.

*Case 2.* Let us now consider the nonhomogeneous case (3.9). By the hypotheses, there exists a $C^2$ function $\psi$ satisfying $\mathcal{N}_s \psi = g$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

Let $\bar u = u - \psi$. Then, $\bar u$ solves

$$\begin{cases} (-\Delta)^s \bar u &= \bar f \quad \text{in } \Omega \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

with

$$\bar f = f - (-\Delta)^s \psi.$$

Then, as we already proved, this problem admits a solution if and only if $\int_\Omega \bar f = 0$, i.e., if

$$0 = \int_\Omega \bar f = \int_\Omega f - \int_\Omega (-\Delta)^s \psi. \tag{3.18}$$

But, by Lemma 3.2, we have that

$$\int_\Omega (-\Delta)^s \psi = -\int_{\mathbb{R}^n\setminus\Omega} \mathcal{N}_s \psi = -\int_{\mathbb{R}^n\setminus\Omega} g.$$

From this and (3.18) we conclude that a solution exists if and only if (3.11) holds.

Finally, the solution is unique up to an additive constant thanks to Lemma 3.8. □

3.3. **Eigenvalues and eigenfunctions.** Here we discuss the spectral properties of problem (1.1). For it, we will need the following classical tool.

**Lemma 3.10** (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $s \in (0,1)$. Then, for all functions $u \in H^s(\Omega)$, we have*

$$\int_\Omega \left| u - \fint_\Omega u \right|^2 dx \le C_\Omega \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy,$$

*where the constant $C_\Omega > 0$ depends only on $\Omega$ and $s$.*

*Proof.* We give the details for the facility of the reader. We argue by contradiction and we assume that the inequality does not hold. Then, there exists a sequence of functions $u_k \in H^s(\Omega)$ satisfying

$$\fint_\Omega u_k = 0, \qquad \|u_k\|_{L^2(\Omega)} = 1, \tag{3.19}$$

and

$$\int_\Omega \int_\Omega \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < \frac{1}{k}. \tag{3.20}$$

In particular, the functions $\{u_k\}_{k\ge1}$ are bounded in $H^s(\Omega)$.

Using now that the embedding $H^s(\Omega) \subset L^2(\Omega)$ is compact (see e.g. [12]), it follows that a subsequence $\{u_{k_j}\}_{j\ge1}$ converges to a function $\bar{u} \in L^2(\Omega)$, i.e.,

$$u_{k_j} \longrightarrow \bar{u} \quad \text{in } L^2(\Omega).$$

Moreover, we deduce from (3.19) that

$$\fint_\Omega \bar{u} = 0, \qquad \text{and} \qquad \|\bar{u}\|_{L^2(\Omega)} = 1. \tag{3.21}$$

On the other hand, (3.20) implies that

$$\int_\Omega \int_\Omega \frac{|\bar{u}(x)-\bar{u}(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = 0.$$

Thus, $\bar{u}$ is constant in $\Omega$, and this contradicts (3.21). □

We finally give the description of the eigenvalues of $(-\Delta)^s$ with zero Neumann boundary conditions.

**Theorem 3.11.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and consider the eigenvalue problem*

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

*Then, there exists a sequence of nonnegative eigenvalues*

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots,$$

*and its corresponding eigenfunctions are a complete orthogonal system in $L^2(\Omega)$.*

*Proof.* We define

$$L^2_0(\Omega) := \left\{ u \in L^2(\Omega) \; : \; \int_\Omega u = 0 \right\}.$$

Let $T : L^2_0(\Omega) \longrightarrow L^2_0(\Omega)$ be defined by $Tf = u$, where $u$ is the unique solution of

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega} \end{cases}$$

according to Definition 3.6.

We remark that the existence and uniqueness of such solution is a consequence of the fact that $f \in L^2_0(\Omega)$ and Theorem 3.9.

Also, we claim that the operator $T$ is compact and self-adjoint.

We first show that $T$ is compact. Indeed, taking $v = u$ in the weak formulation of the problem (3.10), we obtain

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \le \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \tag{3.22}$$

Now, using the Poincaré inequality in Lemma 3.10 (recall that $⨍_\Omega u = 0$), we deduce that

$$\|u\|_{L^2(\Omega)} \le C \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2}. \tag{3.23}$$

This and (3.22) give that

$$\left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{1/2} \le C \|f\|_{L^2(\Omega)}. \tag{3.24}$$

Now, we take a sequence $\{f_k\}_{k \in \mathbb{N}}$ bounded in $L^2(\Omega)$. From (3.23) and (3.24) we obtain that $u_k = Tf_k$ is bounded in $H^s(\Omega)$. Hence, since the embedding $H^s(\Omega) \subset L^2(\Omega)$ is compact, there exists a subsequence that converges in $L^2(\Omega)$. Therefore, $T$ is compact.

Now we show that $T$ is self-adjoint in $L^2_0(\Omega)$. For this, we take $f_1$ and $f_2$ in $C^\infty_0(\Omega)$, with $⨍_\Omega f_1 = ⨍_\Omega f_2 = 0$. Then from the weak formulation in (3.10) we have that, for every $v, w \in H^s_\Omega$,

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{(Tf_1(x) - Tf_1(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy = \int_\Omega f_1 \, v \tag{3.25}$$

and

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{(Tf_2(x) - Tf_2(y))(w(x) - w(y))}{|x-y|^{n+2s}} \, dx \, dy = \int_\Omega f_2 \, w. \tag{3.26}$$

We observe that we can take $v := Tf_2$ in (3.25) and $w := Tf_1$ in (3.26), obtaining that

$$\int_\Omega f_1 \, Tf_2 = \int_\Omega f_2 \, Tf_1, \quad \text{for any } f_1, f_2 \in C^\infty_0(\Omega). \tag{3.27}$$

Now, if $f_1, f_2 \in L^2_0(\Omega)$ we can find sequences of functions $f_{1,k}, f_{2,k} \in C^\infty_0(\Omega)$ such that $f_{1,k} \to f_1$ and $f_{2,k} \to f_2$ in $L^2(\Omega)$ as $k \to +\infty$. Therefore, from (3.27), we have

$$\int_\Omega f_{1,k} \, Tf_{2,k} = \int_\Omega f_{2,k} \, Tf_{1,k}. \tag{3.28}$$

We notice that, thanks to (3.23) and (3.24), $Tf_{1,k} \to Tf_1$ and $Tf_{2,k} \to Tf_2$ in $L^2(\Omega)$ as $k \to +\infty$, and therefore, from (3.28), we obtain that

$$\int_\Omega f_1\, Tf_2 = \int_\Omega f_2\, Tf_1,$$

thus proving that $T$ is self-adjoint in $L^2_0(\Omega)$.

Thus, by the spectral theorem there exists a sequence of eigenvalues $-\infty < \lambda_2 \le \lambda_3 \le \cdots$, and its corresponding eigenfunctions are a complete orthogonal system in $L^2_0(\Omega)$.

We notice that $\lambda_2 > 0$. Indeed, its corresponding eigenfunction $u_2$ solves

$$\begin{cases} (-\Delta)^s u_2 &= \lambda_2 u_2 \quad \text{in } \Omega \\ \mathcal{N}_s u_2 &= 0 \qquad\;\; \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{3.29}$$

Then, if we take $u_2$ as a test function in the weak formulation of (3.29), we obtain that

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{n+2s}}\, dx\, dy = \lambda_2 \int_\Omega u_2^2,$$

which implies that $\lambda_2 \ge 0$. Now, suppose by contradiction that $\lambda_2 = 0$. Then, from Lemma 3.8 we have that $u_2$ is constant. On the other hand, we know that $u_2 \in L^2_0(\Omega)$, and this implies that $u_2 = 0$, which is a contradiction since $u_2$ is an eigenfunction.

Now, we notice that $\lambda_1 = 0$ is an eigenvalue (its eigenfunction is a constant), thanks to Lemma 3.8. Therefore, we have a sequence of eigenvalues $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$, and its corresponding eigenfunctions are a complete orthogonal system in $L^2(\Omega)$. This concludes the proof. □

In the following proposition we deal with the behavior of the solution of (1.1) at infinity.

**Proposition 3.12.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u \in H^s_\Omega$ be a weak solution (according to Definition 3.6) of*

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ \mathcal{N}_s u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

*Then*

$$\lim_{|x|\to\infty} u(x) = \frac{1}{|\Omega|}\int_\Omega u \quad \text{uniformly in } x.$$

*Proof.* First we observe that, since $\Omega$ is bounded, there exists $R > 0$ such that $\Omega \subset B_R$. Hence, if $y \in \Omega$, we have that

$$|x| - R \le |x-y| \le |x| + R,$$

and so

$$1 - \frac{R}{|x|} \le \frac{|x-y|}{|x|} \le 1 + \frac{R}{|x|}.$$

Therefore, given $\epsilon > 0$, there exists $\bar{R} > R$ such that, for any $|x| \ge \bar{R}$, we have

$$\frac{|x|^{n+2s}}{|x-y|^{n+2s}} = 1 + \gamma(x,y),$$

where $|\gamma(x,y)| \le \epsilon$.

Recalling the definition of $\mathcal{N}_s u$ given in (1.2) and using the fact that $\mathcal{N}_s u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, we have that for any $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$
\begin{aligned}
u(x) &= \frac{\displaystyle\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} = \frac{\displaystyle\int_\Omega \frac{|x|^{n+2s}u(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{|x|^{n+2s}}{|x-y|^{n+2s}}\,dy} \\
&= \frac{\displaystyle\int_\Omega (1+\gamma(x,y))\,u(y)\,dy}{\displaystyle\int_\Omega (1+\gamma(x,y))\,dy} \\
&= \frac{\displaystyle\int_\Omega u(y)\,dy + \int_\Omega \gamma(x,y)\,u(y)\,dy}{\displaystyle|\Omega| + \int_\Omega \gamma(x,y)\,dy}.
\end{aligned}
$$

We set

$$
\gamma_1(x) := \fint_\Omega \gamma(x,y)\,u(y)\,dy \quad \text{and} \quad \gamma_2(x) := \fint_\Omega \gamma(x,y)\,dy,
$$

and we notice that $|\gamma_1(x)| \le C\epsilon$ and $|\gamma_2(x)| \le \epsilon$, for some $C > 0$.

Hence, we have that for any $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$
\begin{aligned}
\left| u(x) - \fint_\Omega u(y)\,dy \right| &= \left| \frac{\displaystyle\fint_\Omega u(y)\,dy + \gamma_1(x)}{1+\gamma_2(x)} - \fint_\Omega u(y)\,dy \right| \\
&= \frac{\left| \gamma_1(x) - \gamma_2(x) \displaystyle\fint_\Omega u(y)\,dy \right|}{1+\gamma_2(x)} \\
&\le \frac{C\,\epsilon}{1-\epsilon}.
\end{aligned}
$$

Therefore, sending $\epsilon \to 0$ (that is, $|x| \to +\infty$), we obtain the desired result. □

*Remark* 3.13 (Interior regularity of solutions). We notice that, in particular, Proposition 3.12 implies that $u$ is bounded at infinity. Thus, if solutions are locally bounded, then one could apply interior regularity results for solutions to $(-\Delta)^s u = f$ in $\Omega$ (see e.g. [17, 20, 7, 19]).

## 4. The heat equation

Here we show that solutions of the nonlocal heat equation with zero Neumann datum preserve their mass and have energy that decreases in time.

To avoid technicalities, we assume that $u$ is a classical solution of problem (1.4), so that we can differentiate under the integral sign.

**Proposition 4.1.** *Assume that $u(x,t)$ is a classical solution to* (1.4)*, in the sense that $u$ is bounded and $|u_t| + |(-\Delta)^s u| \le K$ for all $t > 0$. Then, for all $t > 0$,*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)dx.$$

*In other words, the total mass is conserved.*

*Proof.* By the dominated convergence theorem, and using Lemma 3.2, we have

$$\frac{d}{dt}\int_\Omega u = \int_\Omega u_t = -\int_\Omega (-\Delta)^s u = \int_{\mathbb{R}^n\setminus\Omega} \mathcal{N}_s u = 0.$$

Thus, the quantity $\int_\Omega u$ does not depend on $t$, and the result follows. □

**Proposition 4.2.** *Assume that $u(x,t)$ is a classical solution to* (1.4)*, in the sense that $u$ is bounded and $|u_t| + |(-\Delta)^s u| \le K$ for all $t > 0$. Then, the energy*

$$E(t) = \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{n+2s}}\,dx\,dy$$

*is decreasing in time $t > 0$.*

*Proof.* Let us compute $E'(t)$, and we will see that it is negative. Indeed, using Lemma 3.3,

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{2\big(u(x,t)-u(y,t)\big)\big(u_t(x,t)-u_t(y,t)\big)}{|x-y|^{n+2s}}\,dx\,dy \\ &= \frac{4}{c_{n,s}}\int_\Omega u_t\,(-\Delta)^s u\,dx, \end{aligned}$$

where we have used that $\mathcal{N}_s u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

Thus, using now the equation $u_t + (-\Delta)^s u = 0$ in $\Omega$, we find

$$E'(t) = -\frac{4}{c_{n,s}}\int_\Omega |(-\Delta)^s u|^2 dx \le 0,$$

with strict inequality unless $u$ is constant. □

Next we prove that solutions of the nonlocal heat equation with Neumann condition approach a constant as $t \to +\infty$:

**Proposition 4.3.** *Assume that $u(x,t)$ is a classical solution to* (1.4)*, in the sense that $u$ is bounded and $|u_t| + |(-\Delta)^s u| \le K$ for all $t > 0$. Then,*

$$u \longrightarrow \frac{1}{|\Omega|}\int_\Omega u_0 \quad \text{in } L^2(\Omega)$$

*as $t \to +\infty$.*

*Proof.* Let

$$m := \frac{1}{|\Omega|}\int_\Omega u_0$$

be the total mass of $u$. Define also

$$A(t) := \int_\Omega |u - m|^2 \, dx.$$

Notice that, by Proposition 4.1, we have

$$A(t) = \int_\Omega \big(u^2 - 2mu + m^2\big) dx = \int_\Omega u^2 \, dx - |\Omega| m^2.$$

Then, by Lemma 3.3,

$$A'(t) = 2 \int_\Omega u_t u \, dx = -2 \int_\Omega u(-\Delta)^s u \, dx = -c_{n,s} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x,t) - u(y,t)|^2}{|x-y|^{n+2s}} \, dx \, dy.$$

Hence, $A$ is decreasing.

Moreover, using the Poincaré inequality in Lemma 3.10 and again Proposition 4.1, we deduce that

$$A'(t) \le -c \int_\Omega |u - m|^2 \, dx = -c \, A(t),$$

for some $c > 0$. Thus, it follows that

$$A(t) \le e^{-ct} A(0),$$

and thus

$$\lim_{t \to +\infty} \int_\Omega |u(x,t) - m|^2 dx = 0,$$

i.e., $u$ converges to $m$ in $L^2(\Omega)$.

Notice that, in fact, we have proved that the convergence is exponentially fast. □

## 5. Limits

In this section we study the limits as $s \to 1$ and the continuity properties induced by the fractional Neumann condition.

### 5.1. Limit as $s \to 1$.

**Proposition 5.1.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain. Let $u$ and $v$ be $C^2_0(\mathbb{R}^n)$ functions. Then,*

$$\lim_{s \to 1} \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u \, v = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, v.$$

*Proof.* By Lemma 3.3, we have that

$$\int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u \, v = \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy - \int_\Omega v(-\Delta)^s u. \tag{5.1}$$

Now, we claim that

$$\lim_{s \to 1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy = \int_\Omega \nabla u \cdot \nabla v. \tag{5.2}$$

We observe that to show (5.2), it is enough to prove that, for any $u \in C^2_0(\mathbb{R}^n)$,

$$\lim_{s \to 1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy = \int_\Omega |\nabla u|^2. \tag{5.3}$$

Indeed,

$$\begin{aligned}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} & \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &= \frac{1}{2}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|(u+v)(x)-(u+v)(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Now, we recall that

$$\lim_{s\to 1}\frac{c_{n,s}}{1-s} = \frac{4n}{\omega_{n-1}},$$

(see Corollary 4.2 in [12]), and so we have to show that

$$\lim_{s\to 1}(1-s)\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \frac{\omega_{n-1}}{2n}\int_\Omega |\nabla u|^2. \tag{5.4}$$

For this, we first show that

$$\lim_{s\to 1}(1-s)\int_{\Omega\times(\mathcal{C}\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = 0. \tag{5.5}$$

Without loss of generality, we can suppose that $B_r\subset\Omega\subset B_R$, for some $0<r<R$. Since $u\in C_0^2(\mathbb{R}^n)$, then

$$\begin{aligned}\int_{\Omega\times(\mathcal{C}\Omega)} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\le 4\|u\|^2_{L^\infty(\mathbb{R}^n)}\int_{\Omega\times(\mathcal{C}\Omega)}\frac{1}{|x-y|^{n+2s}}\,dx\,dy \\ &\le 4\|u\|^2_{L^\infty(\mathbb{R}^n)}\int_{B_R\times(\mathcal{C}B_r)}\frac{1}{|x-y|^{n+2s}}\,dx\,dy \\ &\le 4\|u\|^2_{L^\infty(\mathbb{R}^n)}\,\omega_{n-1}\int_{B_R}dx\int_r^{+\infty}\rho^{n-1}\rho^{-n-2s}\,d\rho \\ &= 4\|u\|^2_{L^\infty(\mathbb{R}^n)}\,\omega_{n-1}\int_{B_R}dx\int_r^{+\infty}\rho^{-1-2s}\,d\rho \\ &= 4\|u\|^2_{L^\infty(\mathbb{R}^n)}\,\frac{\omega_{n-1}\,r^{-2s}}{2s}\int_{B_R}dx \\ &= 4\|u\|^2_{L^\infty(\mathbb{R}^n)}\,\frac{\omega_{n-1}^2\,R^n\,r^{-2s}}{2s},\end{aligned}$$

which implies (5.5). Hence,

$$\begin{aligned}\lim_{s\to 1}(1-s)&\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\ &= \lim_{s\to 1}(1-s)\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = C_n\int_\Omega|\nabla u|^2,\end{aligned} \tag{5.6}$$

where $C_n>0$ depends only on the dimension, see [5].

In order to determine the constant $C_n$, we take a $C^2$-function $u$ supported in $\Omega$. In this case, we have

$$\int_\Omega |\nabla u|^2\,dx = \int_{\mathbb{R}^n} |\nabla u|^2\,dx = \int_{\mathbb{R}^n} |\xi|^2\,|\hat{u}(\xi)|^2\,d\xi, \tag{5.7}$$

where $\hat{u}$ is the Fourier transform of $u$. Moreover,

$$\begin{aligned}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &= \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= 2\,c_{n,s}^{-1}\int_{\mathbb{R}^n} |\xi|^{2s}\,|\hat{u}(\xi)|^2\,dx,\end{aligned}$$

thanks to Proposition 3.4 in [12]. Therefore, using Corollary 4.2 in [12] and (5.7), we have

$$\begin{aligned}&\lim_{s\to1}(1-s)\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= \lim_{s\to1}\frac{2(1-s)}{c_{n,s}}\int_{\mathbb{R}^n}|\xi|^{2s}\,|\hat{u}(\xi)|^2\,dx\\ &= \frac{\omega_{n-1}}{2n}\int_{\mathbb{R}^n}|\xi|^2\,|\hat{u}(\xi)|^2\,dx\\ &= \frac{\omega_{n-1}}{2n}\int_\Omega|\nabla u|^2\,dx.\end{aligned}$$

Hence, the constant in (5.6) is $C_n = \frac{\omega_{n-1}}{2n}$. This concludes the proof of (5.4), and in turn of (5.2).

On the other hand,

$$-(-\Delta)^s u \to \Delta u \qquad \text{uniformly in } \mathbb{R}^n,$$

(see Proposition 4.4 in [12]). This, (5.1) and (5.2) give

$$\lim_{s\to1}\int_{\mathbb{R}^n\setminus\Omega}\mathcal{N}_s u\,v = \int_\Omega \nabla u\cdot\nabla v + \int_\Omega v\,\Delta u = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,v,$$

as desired. □

5.2. **Continuity properties.** Following is a continuity result for functions satisfying the nonlocal Neumann condition:

**Proposition 5.2.** *Let $\Omega\subset\mathbb{R}^n$ be a domain with $C^1$ boundary. Let $u$ be continuous in $\overline{\Omega}$, with $\mathcal{N}_s u = 0$ in $\mathbb{R}^n\setminus\overline{\Omega}$. Then $u$ is continuous in the whole of $\mathbb{R}^n$.*

*Proof.* First, let us fix $x_0\in\mathbb{R}^n\setminus\overline{\Omega}$. Since the latter is an open set, there exists $\rho>0$ such that $|x_0-y|\geq\rho$ for any $y\in\Omega$. Thus, if $x\in B_{\rho/2}(x_0)$, we have that $|x-y|\geq |x_0-y|-|x_0-x|\geq\rho/2$.

Moreover, if $x\in B_{\rho/2}(x_0)$, we have that

$$|x-y|\geq|y|-|x_0|-|x_0-x|\geq\frac{|y|}{2}+\left(\frac{|y|}{4}-|x_0|\right)+\left(\frac{|y|}{4}-\frac{\rho}{2}\right)\geq\frac{|y|}{2},$$

provided that $|y|\geq R:=4|x_0|+2\rho$. As a consequence, for any $x\in B_{\rho/2}(x_0)$, we have that

$$\frac{|u(y)|+1}{|x-y|^{n+2s}}\leq 2^{n+2s}\left(\|u\|_{L^\infty(\overline{\Omega})}+1\right)\left(\frac{\chi_{B_R}(y)}{\rho^{n+2s}}+\frac{\chi_{\mathbb{R}^n\setminus B_R}(y)}{|y|^{n+2s}}\right)=:\psi(y)$$

and the function $\psi$ belongs to $L^1(\mathbb{R}^n)$. Thus, by the Neumann condition and the Dominated Convergence Theorem, we obtain that

$$\lim_{x\to x_0} u(x) = \lim_{x\to x_0} \frac{\displaystyle\int_\Omega \frac{u(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} = \frac{\displaystyle\int_\Omega \frac{u(y)}{|x_0-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x_0-y|^{n+2s}}} = u(x_0).$$

This proves that $u$ is continuous at any points of $\mathbb{R}^n \setminus \overline{\Omega}$.

Now we show the continuity at a point $p \in \partial\Omega$. We take a sequence $p_k \to p$ as $k \to +\infty$. We let $q_k$ be the projection of $p_k$ to $\overline{\Omega}$. Since $p \in \overline{\Omega}$, we have from the minimizing property of the projection that

$$|p_k - q_k| = \inf_{\xi\in\overline{\Omega}} |p_k - \xi| \le |p_k - p|,$$

and so

$$|q_k - p| \le |q_k - p_k| + |p_k - p| \le 2|p_k - p| \to 0$$

as $k \to +\infty$. Therefore, since we already know from the assumptions the continuity of $u$ at $\overline{\Omega}$, we obtain that

$$\lim_{k\to+\infty} u(q_k) = u(p). \tag{5.8}$$

Now we claim that

$$\lim_{k\to+\infty} u(p_k) - u(q_k) = 0. \tag{5.9}$$

To prove it, it is enough to consider the points of the sequence $p_k$ that belong to $\mathbb{R}^n \setminus \overline{\Omega}$ (since, of course, the points $p_k$ belonging to $\overline{\Omega}$ satisfy $p_k = q_k$ and for them (5.9) is obvious). We define $\nu_k := (p_k - q_k)/|p_k - q_k|$. Notice that $\nu_k$ is the exterior normal of $\Omega$ at $q_k \in \partial\Omega$. We consider a rigid motion $\mathcal{R}_k$ such that $\mathcal{R}_k q_k = 0$ and $\mathcal{R}_k \nu_k = e_n = (0, \dots, 0, 1)$. Let also $h_k := |p_k - q_k|$. Notice that

$$h_k^{-1}\mathcal{R}_k p_k = h_k^{-1}\mathcal{R}_k(p_k - q_k) = \mathcal{R}_k \nu_k = e_n. \tag{5.10}$$

Then, the domain

$$\Omega_k := h_k^{-1}\mathcal{R}_k \Omega$$

has vertical exterior normal at 0 and approaches the halfspace $\Pi := \{x_n < 0\}$ as $k \to +\infty$.

Now, we use the Neumann condition at $p_k$ and we obtain that

$$\begin{aligned} u(p_k) - u(q_k) &= \frac{\displaystyle\int_\Omega \frac{u(y)}{|p_k-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|p_k-y|^{n+2s}}} - u(q_k) \\ &= \frac{\displaystyle\int_\Omega \frac{u(y)-u(q_k)}{|p_k-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|p_k-y|^{n+2s}}} = I_1 + I_2, \end{aligned}$$

with

$$I_1 := \frac{\displaystyle\int_{\Omega\cap B_{\sqrt{h_k}}(q_k)} \frac{u(y)-u(q_k)}{|p_k-y|^{n+2s}}\,dy}{\displaystyle\int_{\Omega} \frac{dy}{|p_k-y|^{n+2s}}}$$

$$\text{and} \qquad I_2 := \frac{\displaystyle\int_{\Omega\setminus B_{\sqrt{h_k}}(q_k)} \frac{u(y)-u(q_k)}{|p_k-y|^{n+2s}}\,dy}{\displaystyle\int_{\Omega} \frac{dy}{|p_k-y|^{n+2s}}}.$$

We observe that the uniform continuity of $u$ in $\overline{\Omega}$ gives that

$$\lim_{k\to+\infty} \sup_{y\in\Omega\cap B_{\sqrt{h_k}}(q_k)} |u(y)-u(q_k)| = 0.$$

As a consequence

$$|I_1| \le \sup_{y\in\Omega\cap B_{\sqrt{h_k}}(q_k)} |u(y)-u(q_k)| \to 0 \tag{5.11}$$

as $k\to+\infty$. Moreover, exploiting the change of variable $\eta := h_k^{-1}\mathcal{R}_k y$ and recalling (5.10), we obtain that

$$\begin{aligned}
|I_2| &\le \frac{\displaystyle\int_{\Omega\setminus B_{\sqrt{h_k}}(q_k)} \frac{|u(y)-u(q_k)|}{|p_k-y|^{n+2s}}\,dy}{\displaystyle\int_{\Omega} \frac{dy}{|p_k-y|^{n+2s}}} \\
&\le 2\|u\|_{L^\infty(\overline{\Omega})} \frac{\displaystyle\int_{\Omega\setminus B_{\sqrt{h_k}}(q_k)} \frac{dy}{|p_k-y|^{n+2s}}}{\displaystyle\int_{\Omega} \frac{dy}{|p_k-y|^{n+2s}}} \\
&= 2\|u\|_{L^\infty(\overline{\Omega})} \frac{\displaystyle\int_{\Omega_k\setminus B_{1/\sqrt{h_k}}} \frac{d\eta}{|e_n-\eta|^{n+2s}}}{\displaystyle\int_{\Omega_k} \frac{d\eta}{|e_n-\eta|^{n+2s}}}.
\end{aligned}$$

Notice that, if $\eta\in\Omega_k\setminus B_{1/\sqrt{h_k}}$ then

$$\begin{aligned}
|e_n-\eta|^{n+2s} &= |e_n-\eta|^{n+s}|e_n-\eta|^{s} \ge |e_n-\eta|^{n+s}\Big(|\eta|-1\Big)^s \\
&\ge |e_n-\eta|^{n+s}\Big(h_k^{-1/2}-1\Big)^s \ge |e_n-\eta|^{n+s} h_k^{-s/4}
\end{aligned}$$

for large $k$. Therefore

$$|I_2| \le 2h_k^{s/4}\|u\|_{L^\infty(\overline{\Omega})} \frac{\displaystyle\int_{\Omega_k} \frac{d\eta}{|e_n-\eta|^{n+s}}\,dy}{\displaystyle\int_{\Omega_k} \frac{d\eta}{|e_n-\eta|^{n+2s}}}.$$

Since

$$\lim_{k\to+\infty} \frac{\displaystyle\int_{\Omega_k} \frac{d\eta}{|e_n-\eta|^{n+s}}\,dy}{\displaystyle\int_{\Omega_k} \frac{d\eta}{|e_n-\eta|^{n+2s}}} = \frac{\displaystyle\int_{\Pi} \frac{d\eta}{|e_n-\eta|^{n+s}}\,dy}{\displaystyle\int_{\Pi} \frac{d\eta}{|e_n-\eta|^{n+2s}}},$$

we conlude that $|I_2| \to 0$ as $k \to +\infty$. This and (5.11) imply (5.9).

From (5.8) and (5.9), we conclude that

$$\lim_{k\to+\infty} u(p_k) = u(p),$$

hence $u$ is continuous at $p$. □

As a direct consequence of Proposition 5.2 we obtain:

**Corollary 5.3.** *Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^1$ boundary. Let $v_0 \in C(\mathbb{R}^n)$. Let*

$$v(x) := \begin{cases} v_0(x) & \text{if } x \in \overline{\Omega}, \\ \dfrac{\displaystyle\int_\Omega \frac{v_0(y)}{|x-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x-y|^{n+2s}}} & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

*Then $v \in C(\mathbb{R}^n)$ and it satisfies $v = v_0$ in $\overline{\Omega}$ and $\mathcal{N}_s v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$.*

*Proof.* By construction, $v = v_0$ in $\overline{\Omega}$ and $\mathcal{N}_s v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Then we can use Proposition 5.2 and obtain that $v \in C(\mathbb{R}^n)$. □

Now we study the boundary behavior of the nonlocal Neumann function $\tilde{\mathcal{N}}_s u$.

**Proposition 5.4.** *Let $\Omega \subset \mathbb{R}^n$ be a $C^1$ domain, and $u \in C(\mathbb{R}^n)$. Then, for all $s \in (0, 1)$,*

$$\lim_{\substack{x\to\partial\Omega \\ x\in\mathbb{R}^n\setminus\overline{\Omega}}} \tilde{\mathcal{N}}_s u(x) = 0. \tag{5.12}$$

*Also, if $s > \frac{1}{2}$ and $u \in C^{1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0, 2s-1)$, then*

$$\partial_\nu \tilde{\mathcal{N}}_s u(x) := \lim_{\epsilon\to0^+} \frac{\tilde{\mathcal{N}}_s u(x+\epsilon\nu)}{\epsilon} = \kappa\, \partial_\nu u \qquad \text{for any } x \in \partial\Omega, \tag{5.13}$$

*for some constant $\kappa > 0$.*

*Proof.* Let $x_k$ be a sequence in $\mathbb{R}^n \setminus \overline{\Omega}$ such that $x_k \to x_\infty \in \partial\Omega$ as $k \to +\infty$.

By Corollary 5.3 (applied here with $v_0 := u$), there exists $v \in C(\mathbb{R}^n)$ such that $v = u$ in $\overline{\Omega}$ and $\mathcal{N}_s v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. By the continuity of $u$ and $v$ we have that

$$\lim_{k\to+\infty} u(x_k) - v(x_k) = u(x_\infty) - v(x_\infty) = 0. \tag{5.14}$$

Moreover

$$\begin{aligned}
\tilde{\mathcal{N}}_s u(x_k) &= \tilde{\mathcal{N}}_s u(x_k) - \tilde{\mathcal{N}}_s v(x_k) \\
&= \frac{\displaystyle\int_\Omega \frac{u(x_k)-u(y)}{|x_k-y|^{n+2s}}\,dy - \int_\Omega \frac{v(x_k)-v(y)}{|x_k-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x_k-y|^{n+2s}}} \\
&= \frac{\displaystyle\int_\Omega \frac{u(x_k)-v(x_k)}{|x_k-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|x_k-y|^{n+2s}}} \\
&= u(x_k) - v(x_k).
\end{aligned}$$

This and (5.14) imply that

$$\lim_{k\to+\infty} \tilde{\mathcal{N}}_s u(x_k) = 0,$$

that is (5.12).

Now, we prove (5.13). For this, we suppose that $s > \frac{1}{2}$, that $0 \in \partial\Omega$ and that the exterior normal $\nu$ coincides with $e_n = (0,\dots,0,1)$; then we use (5.12) and the change of variable $\eta := \epsilon^{-1} y$ in the following computation:

$$\begin{aligned}
\epsilon^{-1}\Big(\tilde{\mathcal{N}}_s u(\epsilon e_n) - \tilde{\mathcal{N}}_s u(0)\Big) &= \epsilon^{-1}\tilde{\mathcal{N}}_s u(\epsilon e_n) \\
&= \frac{\displaystyle\epsilon^{-1}\int_\Omega \frac{u(\epsilon e_n)-u(y)}{|\epsilon e_n-y|^{n+2s}}\,dy}{\displaystyle\int_\Omega \frac{dy}{|\epsilon e_n-y|^{n+2s}}} \\
&= \frac{\displaystyle\epsilon^{-1}\int_{\frac{1}{\epsilon}\Omega} \frac{u(\epsilon e_n)-u(\epsilon\eta)}{|e_n-\eta|^{n+2s}}\,d\eta}{\displaystyle\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n-\eta|^{n+2s}}} = I_1 + I_2,
\end{aligned}$$

where

$$I_1 := \frac{\displaystyle\int_{\frac{1}{\epsilon}\Omega} \frac{\nabla u(\epsilon e_n)\cdot(e_n-\eta)}{|e_n-\eta|^{n+2s}}\,d\eta}{\displaystyle\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n-\eta|^{n+2s}}}$$

$$\text{and} \qquad I_2 := \frac{\displaystyle\epsilon^{-1}\int_{\frac{1}{\epsilon}\Omega} \frac{u(\epsilon e_n)-u(\epsilon\eta)-\epsilon\nabla u(\epsilon e_n)\cdot(e_n-\eta)}{|e_n-\eta|^{n+2s}}\,d\eta}{\displaystyle\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n-\eta|^{n+2s}}}.$$

So, if $\Pi := \{x_n < 0\}$, we have that

$$\begin{aligned}\lim_{\epsilon\to 0^+} I_1 &= \frac{\displaystyle\int_\Pi \frac{\nabla u(0)\cdot(e_n-\eta)}{|e_n-\eta|^{n+2s}}\,d\eta}{\displaystyle\int_\Pi \frac{d\eta}{|e_n-\eta|^{n+2s}}} \\ &= \frac{\displaystyle\int_\Pi \frac{\partial_n u(0)(1-\eta_n)}{|e_n-\eta|^{n+2s}}\,d\eta}{\displaystyle\int_\Pi \frac{d\eta}{|e_n-\eta|^{n+2s}}},\end{aligned}$$

where we have used that, for any $i \in \{1,\dots,n-1\}$ the map $\eta \mapsto \frac{\partial_i u(0)\cdot\eta_i}{|e_n-\eta|^{n+2s}}$ is odd and so its integral averages to zero. So, we can write

$$\lim_{\epsilon\to 0^+} I_1 = \kappa\,\partial_n u(0) \quad\text{with}\quad \kappa := \frac{\displaystyle\int_\Pi \frac{(1-\eta_n)}{|e_n-\eta|^{n+2s}}\,d\eta}{\displaystyle\int_\Pi \frac{d\eta}{|e_n-\eta|^{n+2s}}}. \tag{5.15}$$

We remark that $\kappa$ is finite, since $s > \frac{1}{2}$. Moreover

$$\begin{aligned}&\epsilon^{-1}\Big|u(\epsilon e_n) - u(\epsilon\eta) - \epsilon\nabla u(\epsilon e_n)\cdot(e_n-\eta)\Big| \\ =\;& \left|\int_0^1 \Big(\nabla u(t\epsilon e_n + (1-t)\epsilon\eta) - \nabla u(\epsilon e_n)\Big)\cdot(e_n-\eta)\,dt\right| \\ \le\;& \|u\|_{C^{1,\alpha}(\mathbb{R}^n)}\,|e_n-\eta| \int_0^1 |t\epsilon e_n + (1-t)\epsilon\eta - \epsilon e_n|^\alpha\,dt \\ \le\;& \|u\|_{C^{1,\alpha}(\mathbb{R}^n)}\,\epsilon^\alpha\,|e_n-\eta|^{1+\alpha}.\end{aligned}$$

As a consequence

$$\epsilon^{-\alpha}|I_2| \le \frac{\displaystyle\|u\|_{C^{1,\alpha}(\mathbb{R}^n)}\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n-\eta|^{n+2s-1-\alpha}}\,d\eta}{\displaystyle\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n-\eta|^{n+2s}}} \longrightarrow \frac{\displaystyle\|u\|_{C^{1,\alpha}(\mathbb{R}^n)}\int_\Pi \frac{d\eta}{|e_n-\eta|^{n+2s-1-\alpha}}\,d\eta}{\displaystyle\int_\Pi \frac{d\eta}{|e_n-\eta|^{n+2s}}}$$

as $\epsilon \to 0$, which is finite, thanks to our assumptions on $\alpha$. This shows that $I_2 \to 0$ as $\epsilon \to 0$. Hence, recalling (5.15), we get that

$$\lim_{\epsilon\to 0^+} \epsilon^{-1}\Big(\tilde{\mathcal{N}}_s u(\epsilon e_n) - \tilde{\mathcal{N}}_s u(0)\Big) = \kappa\,\partial_n u(0),$$

which establishes (5.13). □

## 6. An overdetermined problem

In this section we consider an overdetemined problem. For this, we will use the renormalized nonlocal Neumann condition that has been introduced in Remark 3.4. Indeed, as we pointed out in Remark 3.5, this is natural if one considers nonhomogeneous Neumann conditions.

**Theorem 6.1.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and Lipschitz domain. Then there exists no function $u \in C(\mathbb{R}^n)$ satisfying*

$$\begin{cases} u(x) = 0 & \text{for any } x \in \mathbb{R}^n \setminus \Omega \\ \tilde{\mathcal{N}}_s u(x) = 1 & \text{for any } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{6.1}$$

*Remark* 6.2. We notice that $u = \chi_\Omega$ satisfies (6.1), but it is a discontinuous function.

*Proof.* Without loss of generality, we can suppose that $0 \in \partial\Omega$. We argue by contradiction and we assume that there exists a continuous function $u$ that satisfies (6.1). Therefore, there exists $\delta > 0$ such that

$$|u| \le 1/2 \text{ in } B_\delta. \tag{6.2}$$

Since $\Omega$ is Lipschitz, up to choosing $\delta$ small enough, we have that $\Omega \cap B_\delta = \tilde{\Omega} \cap B_\delta$, where

$$\tilde{\Omega} := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n < \gamma(x')\}$$

for a suitable Lipschitz function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\gamma(0) = 0$ and $\partial_{x'}\gamma(0) = 0$.

Now we let $x := \epsilon\, e_n \in \mathbb{R}^n \setminus \overline{\Omega}$, for suitable $\epsilon > 0$ sufficiently small. We observe that

$$u(\epsilon\, e_n) = 0. \tag{6.3}$$

Moreover we consider the set

$$\frac{1}{\epsilon}\tilde{\Omega} = \left\{ y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } y_n < \frac{1}{\epsilon}\gamma(\epsilon y') \right\}.$$

We also define

$$K := \left\{ y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } y_n < -L\,|y'| \right\},$$

where $L$ is the Lipschitz constant of $\gamma$.

We claim that

$$K \subseteq \epsilon^{-1}\,\tilde{\Omega}. \tag{6.4}$$

Indeed, since $\gamma$ is Lipschitz and $0 \in \partial\Omega$, we have that

$$-\gamma(\epsilon y') = -\gamma(\epsilon y') + \gamma(0) \le L\,\epsilon\,|y'|,$$

and so, if $y \in K$,

$$y_n \le -L\,|y'| \le \frac{1}{\epsilon}\gamma(\epsilon y'),$$

which implies that $y \in \epsilon^{-1}\tilde{\Omega}$. This shows (6.4).

Now we define

$$\Sigma_\epsilon := \int_{B_\delta \cap \Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}},$$

and we observe that

$$\int_{B_\delta \cap \Omega} \frac{u(y) - u(\epsilon\, e_n)}{|\epsilon e_n - y|^{n+2s}}\, dy \le \frac{1}{2}\,\Sigma_\epsilon, \tag{6.5}$$

thanks to (6.3) and (6.2). Furthermore, if $y \in \mathbb{R}^n \setminus B_\delta$ and $\epsilon \le \delta/2$, we have

$$|y - \epsilon e_n| \ge |y| - \epsilon \ge \frac{|y|}{2},$$

which implies that

$$\int_{\Omega \setminus B_\delta} \frac{u(y) - u(\epsilon\, e_n)}{|\epsilon e_n - y|^{n+2s}}\, dy \le C \int_{\Omega \setminus B_\delta} \frac{dy}{|\epsilon e_n - y|^{n+2s}} \le C \int_{\mathbb{R}^n \setminus B_\delta} \frac{dy}{|y|^{n+2s}}\, dy = C\,\delta^{-2s}, \tag{6.6}$$

up to renaming the constants.

On the othe hand, we have that

$$\int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} \geq \int_{B_\delta \cap \Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} = \Sigma_\epsilon. \tag{6.7}$$

Finally, we observe that

$$\begin{aligned} \epsilon^{2s}\,\Sigma_\epsilon &= \epsilon^{2s} \int_{B_\delta \cap \Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} \\ &= \int_{B_{\delta/\epsilon} \cap (\epsilon^{-1}\Omega)} \frac{dz}{|e_n - z|^{n+2s}} \\ &\geq \int_{B_{\delta/\epsilon} \cap K} \frac{dz}{|e_n - z|^{n+2s}} \\ &=: \kappa, \end{aligned} \tag{6.8}$$

where we have used the change of variable $y = \epsilon z$ and (6.4).

Hence, using the second condition in (6.1) and putting together (6.5), (6.6), (6.7) and (6.8), we obtain

$$\begin{aligned} 0 &= \int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} - \int_{\Omega} \frac{u(\epsilon e_n) - u(x)}{|\epsilon e_n - y|^{n+2s}}\,dy \\ &= \int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} - \int_{\Omega \cap B_\delta} \frac{u(\epsilon e_n) - u(x)}{|\epsilon e_n - y|^{n+2s}}\,dy - \int_{\Omega \setminus B_\delta} \frac{u(\epsilon e_n) - u(x)}{|\epsilon e_n - y|^{n+2s}}\,dy \\ &\geq \Sigma_\epsilon - \frac{1}{2}\Sigma_\epsilon - C\,\delta^{-2s} \\ &= \frac{1}{2}\Sigma_\epsilon - C\,\delta^{-2s} \\ &= \epsilon^{-2s}\left(\frac{\epsilon^{2s}}{2}\Sigma_\epsilon - C\,\epsilon^{2s}\,\delta^{-2s}\right) \\ &\geq \epsilon^{-2s}\left(\frac{\kappa}{2} - C\,\epsilon^{2s}\,\delta^{-2s}\right) > 0 \end{aligned}$$

if $\epsilon$ is sufficiently small. This gives a contradiction and concludes the proof. □

## 7. Comparison with previous works

In this last section we compare our new Neumann nonlocal conditions with the previous works in the literature that also deal with Neumann-type conditions for the fractional Laplacian $(-\Delta)^s$.

The idea of [4, 8] (and also [9, 10, 11]) is to consider the *regional* fractional Laplacian, associated to the Dirichlet form

$$c_{n,s} \int_{\Omega} \int_{\Omega} \frac{\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)}{|x - y|^{n+2s}}\,dx\,dy. \tag{7.1}$$

This operator corresponds to a censored process, i.e., a process whose jumps are restricted to be in $\Omega$. The operator can be defined in general domains $\Omega$, and seems to give a natural analogue of homogeneous Neumann condition. However, no nonhomogeneous Neumann conditions can be considered with this model, and the operator depends on the domain $\Omega$.

On the other hand, in [1, 3] the usual diffusion associated to the fractional Laplacian (1.3) was considered inside $\Omega$, and thus the "particle" can jump outside $\Omega$. When it jumps outside $\Omega$, then it is "reflected" or "projected" inside $\Omega$ in a deterministic way. Of course, different types of reflections or projections lead to different Neumann conditions. To appropriately define these reflections, some assumptions on the domain $\Omega$ (like smoothness or convexity) need to be done. In contrast with the regional fractional Laplacian, this problem does not have a variational formulation and everything is done in the context of viscosity solutions.

Finally, in [15] a different Neumann problem for the fractional Laplacian was considered. Solutions to this type of Neumann problems are "large solutions", in the sense that they are not bounded in a neighborhood of $\partial\Omega$. More precisely, it is proved in [15] that the following problem is well-posed

$$\begin{cases} (-\Delta)^s u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\ \partial_\nu\big(u/d^{s-1}\big) &= g \quad \text{on } \partial\Omega, \end{cases}$$

where $d(x)$ is the distance to $\partial\Omega$.

With respect to the existing literature, the new Neumann problems (1.1) and (1.4) that we present here have the following advantages:

- The equation satisfied inside $\Omega$ does not depend on anything (domain, right hand side, etc). Notice that the operator in (1.3) does not depend on the domain $\Omega$, while for instance the regional fractional Laplacian defined in (7.1) depends on $\Omega$.
- The problem can be formulated in general domains, including nonsmooth or even unbounded ones.
- The problem has a variational structure. For instance, solutions to the elliptic problem (1.1) can be found as critical points of the functional

$$\mathcal{E}(u) = \frac{c_{n,s}}{4}\int_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega fu.$$

  We notice that the variational formulation of the problem is the analogue of the case $s = 1$. Also, this allows us to easily prove existence of solutions (whenever the compatibility condition $\int_\Omega f = 0$ is satisfied).
- Solutions to the fractional heat equation (1.4) possess natural properties like conservation of mass inside $\Omega$ or convergence to a constant as $t \to +\infty$.
- Our probabilistic interpretation allows us to formulate problems with nonhomogeneous Neumann conditions $\mathcal{N}_s u = g$ in $\mathbb{R}^n \setminus \overline{\Omega}$, or with mixed Dirichlet and Neumann conditions.
- The formulation of nonlinear equations like $(-\Delta)^s u = f(u)$ in $\Omega$ with Neumann conditions is also clear.

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